Van der Corput lemma for Mittag-Leffler functions

Michael Ruzhansky

Ghent University and Queen Mary University of London

www.ruzhansky.org

9 June 2020

Fractional Calculus, Ghent Analysis & PDE / ZOOM

Joint lecture with Berikbol Torebek

ANALYSIS PDE UGENT

WORKSHOP FRACTIONAL CALCULUS

Think fractional gain integral

One of the central results in harmonic analysis

- Johannes Gaultherus van der Corput, Zahlentheoretische Abschätzungen, *Math. Ann.* 84: 1-2 (1921) 53–79. 100 years! (submitted December 1920)
- **van der Corput first lemma.** Let ϕ be real-valued and smooth on $[a,b]$. If ψ is smooth, ϕ' is monotonic, and $|\phi'(x)| \geq 1$ for all $x \in (a,b)$, then

$$\left| \int_a^b e^{i\lambda\phi(x)} \psi(x) dx \right| \leq C\lambda^{-1}, \quad \lambda > 0.$$

- **van der Corput second lemma.** Let ϕ be real-valued and smooth on $[a,b]$. If $|\phi^{(k)}(x)| \geq 1$, $k \geq 2$ for all $x \in [a,b]$ and ψ is smooth, then

$$\left| \int_a^b e^{i\lambda\phi(x)} \psi(x) dx \right| \leq C\lambda^{-1/k}, \quad \lambda > 0.$$

  Related e.g. to sublevel estimates $\operatorname{meas}\{s \in \operatorname{supp}\phi : |\phi(s)| \leq t\} \leq c_k t^{1/k}$.
- ✎ R., Multidimensional decay in van der Corput lemma, *Studia Math.*, 208 (2012), 1-10. Multidimensional decay (γ depends on geometry of $\phi = 1$)

$$\left| \int_{\mathbb{R}^n} e^{i\lambda\phi(x,\nu)} \psi(x,\nu) dx \right| \leq C\lambda^{-\frac{n}{\gamma}}, \quad \lambda > 0.$$

Plan

Many works: Phong-Stein-Sturm, Carbery-Wright, Carbery-Christ-Wright, ...

Mittag-Leffler function: $$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha > 0, \quad \beta \in \mathbb{R}$$

It generalises the exponential function: $E_{1,1}(z) = e^z$ classical M-L is $E_{\alpha,1}$.

We extend the classical $\int_a^b e^{i\lambda\phi(x)}\psi(x)dx$ by

$$\int_{\mathbb{R}} E_{\alpha,\beta}(i\lambda\phi(x))\,\psi(x)dx, \ 0 < \alpha \leq 1, \ \beta > 0.$$

✎ R+Torebek, Van der Corput lemmas for Mittag-Leffler functions. arXiv:2002.07492, and

$$\int_{\mathbb{R}} E_{\alpha,\beta}(i^{\alpha}\lambda\phi(x))\,\psi(x)dx, \ 0 < \alpha \leq 2, \ \beta > 0.$$

✎ R+Torebek, Van der Corput lemmas for Mittag-Leffler functions. II. α-directions. arXiv:2005.04546

Examples of Mittag-Leffler functions

- $E_{1,1}(z) = \exp(z) = e^z$;
- If $\beta = 1$, we have the classical Mittag-Leffler function $E_{\alpha,1}(z) = \sum\limits_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$, $\alpha > 0$.
- $E_{1/2,1}(z) = e^{z^2}\mathrm{erfc}(-z)$, where $\mathrm{erfc}(z) = 1 - \frac{2}{\sqrt{\pi}} \sum\limits_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!(2n+1)}(z)$.
- for $m = 2, 3, ...,$ we have $E_{1,m}(z) = z^{1-m}\left(e^z - \sum\limits_{k=0}^{m-2} \frac{z^k}{k!}\right)$;
- $E_{2,1}(z) = \cosh\sqrt{z} = \frac{e^{\sqrt{z}}+e^{-\sqrt{z}}}{2}$;
- $E_{2,2}(z) = \frac{\sinh\sqrt{z}}{\sqrt{z}} = \frac{e^{\sqrt{z}}-e^{-\sqrt{z}}}{2\sqrt{z}}$;
- for $m \in \mathbb{N}$, we have $E_{2,2m+1}(z) = z^{-m}\left(\cosh\sqrt{z} - \sum\limits_{k=0}^{m-1} \frac{z^k}{(2k)!}\right)$;
- for $m \in \mathbb{N}$, we have $E_{2,2m}(z) = z^{1/2-m}\left(\sinh\sqrt{z} - \sum\limits_{k=0}^{m-1} \frac{z^{k+1/2}}{(2k+1)!}\right)$;

We consider two types of integrals, for $0 < \alpha \leq 1$, $\beta > 0$:

$$I_{\alpha,\beta}(\lambda) = \int_a^b E_{\alpha,\beta}\left(i\lambda\phi(x)\right)\psi(x)dx,$$

and

$$\mathcal{I}_{\alpha,\beta}(\lambda) = \int_a^b E_{\alpha,\beta}\left(i\lambda(\phi(x)-\phi(a))^{\alpha}\right)\psi(x)dx.$$

The latter $\mathcal{I}_{\alpha,\beta}(\lambda)$ is to also consider the extension of **the non-stationary phase principle:** Let ϕ and ψ be smooth functions such that ψ has compact support in (a,b), and $\phi'(x) \neq 0$ for all $x \in [a,b]$. Then for all $N \geq 0$ we have

$$\left|\int_a^b e^{i\lambda\phi(x)}\psi(x)dx\right| \leq C\lambda^{-N}, \quad \lambda > 0.$$

☞ Important: $\phi(x) = 0$ or $\phi(x) \neq 0$ is important when dealing with $E_{\alpha,\beta}$.

Example for $\int_a^b E_{\alpha,\beta}\left(i\lambda\phi(x)\right)\psi(x)dx$

Consider the time-fractional Schrödinger-type equation

$$\mathcal{D}^{\alpha}_{0+,t}u(t,x) - \lambda\mathcal{D}^{\alpha}_{0+,t}u_{xx}(t,x) + iu_{xx}(t,x) - i\mu u(t,x) = 0,\ t>0,\ x\in\mathbb{R},$$

for $0<\alpha<1$, $\lambda,\mu>0$, with Cauchy data

$$u(0,x)=\psi(x),\ x\in\mathbb{R}.$$

☞ $\mathcal{D}^{\alpha}_{0+,t}u(t,x) = \frac{1}{\Gamma(1-\alpha)}\int_a^t (t-s)^{-\alpha}u_s(s,x)ds$ is Caputo FD.

By using the Fourier transform, we obtain

$$u(t,x) = \int_{\mathbb{R}} e^{ix\xi}E_{\alpha,1}\left(i\frac{\xi^2+\mu}{1+\lambda\xi^2}t^{\alpha}\right)\hat{\psi}(\xi)d\xi$$

If $\psi\in L^1(\mathbb{R})$ and $\hat{\psi}\in L^1(\mathbb{R})$, our results give e.g. the dispersive estimate

$$\|u(t,\cdot)\|_{L^{\infty}(\mathbb{R})} \le C(1+t)^{-\alpha}\|\hat{\psi}\|_{L^1(\mathbb{R})},\ t\ge 0.$$

We first consider general $0<\alpha<1,\ \beta>0.$
The results depend on whether ϕ or ϕ' are non-zero.

- $-\infty\leq a<b\leq+\infty$, $\phi:[a,b]\to\mathbb{R}$ measurable, and $\psi\in L^1[a,b]$.
  If $\operatorname{ess\,inf}_{x\in[a,b]}|\phi(x)|>0$, then $|I_{\alpha,\beta}(\lambda)|\lesssim\frac{1}{1+\lambda},\ \lambda\geq 0$
- $-\infty\leq a<b\leq+\infty$, $\phi\in L^\infty[a,b]$ real-valued differentiable monotonic with $\inf\limits_{x\in[a,b]}|\phi'(x)|>0$, and $\psi\in L^\infty[a,b]$. Then $|I_{\alpha,\beta}(\lambda)|\lesssim\frac{\log(2+\lambda)}{1+\lambda}$
- $-\infty\leq a<b\leq+\infty$. $\phi\in L^\infty[a,b]$ real-valued, $\psi\in L^1[a,b]$. If $\operatorname{ess\,inf}_{x\in[a,b]}|\phi(x)|>0$, then $|I_{1,\beta}(\lambda)|\lesssim\frac{1}{(1+\lambda)^{\beta-1}},\ \beta>1$

☞ **Fractional Euler formula:** Let $\alpha,\beta>0$ and $\phi:[a,b]\to\mathbb{C}$. Then for all $\lambda\in\mathbb{C}$ we have $E_{\alpha,\beta}\left(i\lambda\phi(x)\right)=E_{2\alpha,\beta}\left(-\lambda^2\phi^2(x)\right)+i\lambda\phi(x)E_{2\alpha,\beta+\alpha}\left(-\lambda^2\phi^2(x)\right).$

☞ if $0<\alpha<2$, $\beta\in\mathbb{R}$, μ is such that $\pi\alpha/2<\mu<\min\{\pi,\pi\alpha\}$, then
$|E_{\alpha,\beta}(z)|\leq\frac{C}{1+|z|},\ z\in\mathbb{C},\ \mu\leq|\arg(z)|\leq\pi.$

If $\beta=\alpha$ we have the following improvements:

- $-\infty<a<b<+\infty$. $\phi\in C^1[a,b]$ is real-valued, ϕ' monotonic, $\psi'\in L^1[a,b]$.
  - If $\phi'(x)\neq 0$ for all $x\in[a,b]$, then $|I_{\alpha,\alpha}(\lambda)|\lesssim\frac{1}{1+\lambda}$
  - If $\phi(x)\neq 0$ and $\phi'(x)\neq 0$ for all x then $|I_{\alpha,\alpha}(\lambda)|\lesssim\frac{1}{(1+\lambda)^2},\quad \lambda\geq 0$
- $-\infty<a<b<+\infty$. $\phi\in C^2[a,b]$ is real-valued, $\psi\in C^1[a,b]$.
  - If $\phi'(x)\neq 0$ for all $x\in[a,b]$, then we have $|I_{\alpha,\alpha}(\lambda)|\lesssim\frac{1}{1+\lambda}$
  - If $\phi'(x)\neq 0$ for all x, and $\psi(a)=\psi(b)=0$, then $|I_{\alpha,\alpha}(\lambda)|\lesssim\frac{\log(2+\lambda)}{(1+\lambda)^2}$
  - If $\phi(x)\neq 0$ and $\phi'(x)\neq 0$ for all $x\in[a,b]$, then $|I_{\alpha,\alpha}(\lambda)|\lesssim\frac{1}{(1+\lambda)^2}$

☞ If $\phi'(x)\neq 0$ for all $x\in[a,b]$, Then for any $\alpha>0$ and $\lambda\in\mathbb{C}$ we have

$$E_{\alpha,\alpha}\left(i\lambda\phi(x)\right)=\frac{\alpha}{i\lambda\phi'(x)}\frac{d}{dx}\left(E_{\alpha,1}\left(i\lambda\phi(x)\right)\right).$$

- $-\infty<a<b<+\infty$. ϕ real-valued, $\phi\in C^k[a,b]$, $k\geq 2$, and $\psi'\in L^1[a,b]$. If $|\phi^{(k)}(x)|\geq 1$, for all $x\in[a,b]$, then $|I_{\alpha,\beta}(\lambda)|\lesssim\frac{\log(2+\lambda)}{(1+\lambda)^{\frac{1}{k}}}$
- $-\infty<a<b<+\infty$. ϕ real-valued, $\psi'\in L^1[a,b]$. If $\phi\in C^k[a,b]$, $k\geq 2$, and $|\phi^{(k)}(x)|\geq 1$ for all $x\in[a,b]$, then $|I_{\alpha,\alpha}(\lambda)|\lesssim\frac{1}{(1+\lambda)^{\frac{1}{k}}}$
- **The non-stationary phase principle:** Let $0<\alpha<1$, $-\infty<a<b<+\infty$. Let $\phi\in C^{N+1}[a,b]$ be real-valued increasing, and let $\psi\in C^N[a,b]$, $\psi^{(k)}(a)=0$, $\psi^{(k)}(b)=0$, $k=0,1,...,N-1$, $N\in\mathbb{N}$. If $\phi'(x)\neq 0$ for all $x\in[a,b]$, then for

$$\mathcal{I}_{\alpha,\beta}(\lambda)=\int_a^b E_{\alpha,\beta}\left(i\lambda(\phi(x)-\phi(a))^\alpha\right)\psi(x)dx,$$

we have $|\mathcal{I}_{\alpha,1}(\lambda)|\lesssim\frac{1}{(1+\lambda)^N}$

Let $-\infty<a<b<+\infty$ and $0<\alpha\leq 1/2$, $\beta>2\alpha$. Let $\phi\in L^{\infty}[a,b]$ be a real-valued function and let $\psi\in L^{\infty}[a,b]$.
Suppose that $m_1=\inf_{a\leq x\leq b}|\phi(x)|>0$ and $m_2=\inf_{a\leq x\leq b}|\psi(x)|>0$. Then

$$|I_{\alpha,\beta}(\lambda)|\leq\frac{K_1\|\psi\|_{L^\infty}}{(b-a)}\frac{1+\lambda\|\phi\|_{L^\infty}}{1+k_1\lambda^2m_1^2},\ \lambda\geq 0,$$

where $K_1=\max\left\{\frac{1}{\Gamma(\beta)},\frac{1}{\Gamma(\alpha+\beta)}\right\}$ and $k_1=\min\left\{\frac{\Gamma(\beta)}{\Gamma(2\alpha+\beta)},\frac{\Gamma(\alpha+\beta)}{\Gamma(3\alpha+\beta)}\right\}$.
We also have

$$|I_{\alpha,\beta}(\lambda)|\geq\frac{m_2}{(b-a)\Gamma(\alpha+\beta)}\frac{\lambda m_1}{1+\frac{\Gamma(\beta-\alpha)}{\Gamma(\alpha+\beta)}\lambda^2\|\phi\|^2_{L^\infty}},\ \lambda\geq 0.$$

If $m_1=\inf_{a\leq x\leq b}|\phi(x)|=0$, then

$$|I_{\alpha,\beta}(\lambda)|\geq\frac{m_2}{(b-a)\Gamma(\beta)}\frac{1}{1+\frac{\Gamma(\beta-2\alpha)}{\Gamma(\beta)}\lambda^2\|\phi\|^2_{L^\infty}},\ \ \lambda\geq 0.$$

Sharpness of estimates for $0 < \alpha < 1/2$, $\beta = 2\alpha$

Let $-\infty < a < b < +\infty$ and $0 < \alpha < 1/2$, $\beta = 2\alpha$. Let $\phi \in L^\infty[a,b]$ be a real-valued function and let $\psi \in L^\infty[a,b]$. Let $m_1 = \inf_{a \le x \le b} |\phi(x)| > 0$ and $m_2 = \inf_{a \le x \le b} |\psi(x)| > 0$. Then

$$|I_{\alpha,2\alpha}(\lambda)| \le \frac{K\|\psi\|_{L^\infty}}{(b-a)} \frac{1 + \lambda\|\phi\|_{L^\infty}\left(1 + \sqrt{\frac{\Gamma(1+2\alpha)}{\Gamma(1+4\alpha)}}\lambda^2\|\phi\|_{L^\infty}^2\right)}{\left(1 + \min\left\{\frac{\Gamma(3\alpha)}{\Gamma(5\alpha)}, \sqrt{\frac{\Gamma(1+2\alpha)}{\Gamma(1+4\alpha)}}\right\}\lambda^2 m_1^2\right)^2}, \ \lambda \ge 0,$$

and

$$|I_{\alpha,2\alpha}(\lambda)| \ge \frac{m_2}{(b-a)\Gamma(3\alpha)} \frac{\lambda m_1}{1 + \frac{\Gamma(\alpha)}{\Gamma(3\alpha)}\lambda^2\|\phi\|_{L^\infty}^2}, \ \lambda \ge 0,$$

where $K = \max\left\{\frac{1}{\Gamma(2\alpha)}, \frac{1}{\Gamma(3\alpha)}\right\}$.

If $m_1 = \inf_{a \le x \le b} |\phi(x)| = 0$ and $m_2 = \inf_{a \le x \le b} |\psi(x)| > 0$, then we have

$$|I_{\alpha,2\alpha}(\lambda)| \ge \frac{m_2}{(b-a)\Gamma(2\alpha)} \frac{1}{\left(1 + \sqrt{\frac{\Gamma(1-2\alpha)}{\Gamma(1+2\alpha)}}\lambda^2\|\phi\|_{L^\infty}^2\right)^2}, \ \lambda \ge 0.$$

Generalised Riemann-Lebesgue lemma

Riemann-Lebesgue lemma: if $f \in L^1(a,b)$, then $\lim_{k\to\infty} \int_a^b e^{ikx} f(x)dx = 0.$

If $f \in C^1([a,b])$, van der Corput lemma: $\int_a^b e^{ikx} f(x)dx = \mathcal{O}\left(\frac{1}{k}\right),\ k \to \infty.$

Let $0 < \alpha < 1,\ \beta > 0$ or $\alpha = 1,\ \beta > 1.$

If $f \in L^1(a,b)$, then M-L R-L lemma: $\lim\limits_{k\to\infty} \int_a^b E_{\alpha,\beta}(ikx) f(x)dx = 0.$

If $f \in C^1(a,b)$, then

- for $0 < \alpha < 1, \beta > 0,\ -\infty < a < b < +\infty$, we have $\int_a^b E_{\alpha,\beta}(ikx) f(x)dx = \mathcal{O}\left(\frac{1}{k}\right).$
- for $0 < \alpha < 1, \beta = \alpha,\ 0 < a < b < +\infty$, we have $\int_a^b E_{\alpha,\alpha}(ikx) f(x)dx = \mathcal{O}\left(\frac{1}{k^2}\right).$
- for $\alpha = 1, \beta > 1,\ 0 < a < b < +\infty$, we have $\int_a^b E_{1,\beta}(ikx) f(x)dx = \mathcal{O}\left(\frac{1}{k^{\beta-1}}\right).$

Here we also extend the classical $\int_a^b e^{i\lambda\phi(x)}\psi(x)dx$ Now instead of

$$\int_{\mathbb{R}} E_{\alpha,\beta}\left(i\lambda\phi(x)\right)\psi(x)dx,\ 0<\alpha\leq 1,\ \beta>0,$$

we consider

$$\int_{\mathbb{R}} E_{\alpha,\beta}\left(i^{\alpha}\lambda\phi(x)\right)\psi(x)dx,\ 0<\alpha\leq 2,\ \beta>0.$$

☞ Here we have another useful ingredient (Podlubny, Fractional Differential Equations, Academic Press, 1999): If $0<\alpha<2$, $\beta\in\mathbb{R}$, μ is such that $\pi\alpha/2<\mu<\min\{\pi,\pi\alpha\}$, then

$$|E_{\alpha,\beta}(z)|\leq C_1(1+|z|)^{(1-\beta)/\alpha}\exp(\mathrm{Re}(z^{1/\alpha}))+\frac{C_2}{1+|z|},\ z\in\mathbb{C},\ |\arg(z)|\leq\mu.$$

Examples for $\int_a^b E_{\alpha,\beta}\left(i^\alpha \lambda \phi(x)\right) \psi(x) dx$

Consider the time-fractional Klein-Gordon equation

$$D^\alpha_{0+,t} u(t,x) + i^\alpha u_{xx}(t,x) - i^\alpha \mu u(t,x) = 0, \;\; t > 0, \, x \in \mathbb{R},$$

with $1 < \alpha \leq 2$, $\mu > 0$, and initial data

$$I^{2-\alpha}_{0+,t} u(0,x) = 0, \; \partial_t I^{2-\alpha}_{0+,t} u(0,x) = \psi(x), \; x \in \mathbb{R},$$

with Riemann-Liouville $I^\alpha_{0+,t} u(t,x) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} u(s,x) ds$ and

$D^\alpha_{0+,t} u(t,x) = \frac{1}{\Gamma(2-\alpha)} \partial_t^2 \int\limits_0^t (t-s)^{1-\alpha} u(s,x) ds.$ By Fourier transform:

$$\boxed{u(t,x) = \int\limits_{\mathbb{R}} e^{ix\xi} t^{\alpha-1} E_{\alpha,\alpha}\left(i^\alpha (\xi^2 + \mu) t^\alpha\right) \hat{\psi}(\xi) d\xi}$$

If $\psi, \hat{\psi} \in L^1(\mathbb{R})$ we obtain the dispersive estimate (for all $t > 0$)

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq C t^{\alpha-1} (1+t^\alpha)^{\frac{1-\alpha}{\alpha}} \|\hat{\psi}\|_{L^1(\mathbb{R})} \leq C t^{\alpha-1} (1+t)^{1-\alpha} \|\hat{\psi}\|_{L^1(\mathbb{R})}.$$

Consider the time-fractional Schrödinger equation

$$\mathcal{D}^{\alpha}_{0+,t}u(t,x) + i^{\alpha}u_{xx}(t,x) - i^{\alpha}\mu u(t,x) = t^{-\gamma}\psi(x), \; t>0, \, x \in \mathbb{R},$$

with $0 < \alpha \leq 1,$ $\mu > 0$, $0 \leq \gamma < \alpha$, and Cauchy data

$$u(0,x) = 0, \, x \in \mathbb{R},$$

with $\mathcal{D}^{\alpha}_{0+,t}u(t,x) = \frac{1}{\Gamma(1-\alpha)}\int\limits_0^t (t-s)^{1-\alpha}\, \partial_s u(s,x) ds$ is Caputo FD.

By Fourier transform

$$\boxed{u(t,x) = \Gamma(1-\gamma)\int\limits_{\mathbb{R}} e^{ix\xi} t^{\alpha-\gamma} E_{\alpha,\alpha-\gamma+1}\left(i^{\alpha}(\xi^2+\mu)t^{\alpha}\right)\hat{\psi}(\xi)d\xi}$$

If $\psi, \hat{\psi} \in L^1(\mathbb{R})$, we obtain the dispersive estimate (for all $t > 0$)

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq Ct^{\alpha-\gamma}(1+t^{\alpha})^{\frac{\gamma-\alpha}{\alpha}}\|\hat{\psi}\|_{L^1(\mathbb{R})} \leq Ct^{\alpha-\gamma}(1+t)^{\gamma-\alpha}\|\hat{\psi}\|_{L^1(\mathbb{R})}.$$

☞ Let $\phi:\mathbb{R}\to\mathbb{R}$ be measurable, $\psi\in L^1(\mathbb{R})$. Suppose $m=\operatorname{ess\,inf}_{x\in\mathbb{R}}|\phi(x)|>0$;

(i) for $0<\alpha<2$ and $\beta\geq\alpha+1$ we have

$$|I_{\alpha,\beta}(\lambda)|\leq\frac{M_1}{1+\lambda m}\|\psi\|_{L^1(\mathbb{R})},\ \lambda\geq 1;$$

(ii) for $0<\alpha<2$ and $1<\beta<\alpha+1$ we have

$$|I_{\alpha,\beta}(\lambda)|\leq\frac{M_2}{(1+\lambda m)^{\frac{\beta-1}{\alpha}}}\|\psi\|_{L^1(\mathbb{R})},\ \lambda\geq 1;$$

(iii) for $\alpha=2$ and $\beta>1$ we have

$$|I_{2,\beta}(\lambda)|\leq\frac{M_3}{(1+\lambda m)^{\frac{\beta-1}{2}}}\|\psi\|_{L^1(\mathbb{R})},\ \lambda\geq 1,$$

where M_1,M_2,M_3 above do not depend on ϕ, ψ and λ.

☞ Let $\phi:\mathbb{R}\to\mathbb{R}$ be invertible and differentiable, $\psi\in L^1(\mathbb{R})$. Suppose that $0<\alpha\leq 2$, $\beta=1$, and $m=\inf_{x\in\mathbb{R}}|\phi'(x)|>0$. Then

$$\boxed{|I_{\alpha,1}(\lambda)|\leq\frac{M}{\lambda m}\|\psi\|_{L^1(\mathbb{R})},\ \lambda\geq 1}$$

where M does not depend on ϕ, ψ and λ.

The interval here is finite: $-\infty < a < b < +\infty$. Let ϕ be real-valued, $\phi \in C^k(I)$, $k \geq 1$, and let $\psi \in C^1(I)$. If $|\phi^{(k)}(x)| \geq 1$ for all $x \in I$, then

(i) for $0 < \alpha < 2$ and $\beta \geq \alpha + 1$ we have (M_k independent of λ)

$$|I_{\alpha,\beta}(\lambda)| \leq M_k \left[|\psi(b)| + \int_a^b |\psi'(x)| dx \right] \lambda^{-\frac{1}{k}} \log^{\frac{1}{k}}(1+\lambda), \quad \lambda \geq 1$$

(ii) for $0 < \alpha < 2$ and $1 < \beta < \alpha + 1$ we have (M_k independent of λ)

$$|I_{\alpha,\beta}(\lambda)| \leq M_k \left[|\psi(b)| + \int_a^b |\psi'(x)| dx \right] \lambda^{-\frac{1}{k}} (1+\lambda)^{\frac{\alpha+1-\beta}{\alpha k}}, \quad \lambda \geq 1$$

- If $\beta = \alpha$, we have an improvement (for $k = 1$ and ϕ' monotonic, or $k \geq 2$)

$$|I_{\alpha,\alpha}(\lambda)| \leq M_k \left[|\psi(b)| + \int_a^b |\psi'(x)| dx \right] \lambda^{-1/k}, \quad \lambda \geq 1$$

☞ Let $\phi \in C^2(I)$, $|\phi'(x)| \geq 1$ for all $x \in I$. Then (with M independent of λ)

$$|I_{\alpha,\alpha}(\lambda)| \leq M \left[|\psi(b)| + \int_a^b |\psi'(x)| dx \right] \lambda^{-1}, \quad \lambda \geq 1$$

Generalised Riemann-Lebesgue lemma, type II

Riemann-Lebesgue lemma: if $f \in L^1(a,b)$, then $\lim_{k\to\infty} \int_a^b e^{ikx} f(x)dx = 0.$

If $f \in C^1([a,b])$, van der Corput lemma: $\int_a^b e^{ikx} f(x)dx = \mathcal{O}\left(\frac{1}{k}\right),\ k \to \infty.$

Let $0 < \alpha < 1,\ \beta > 0$ or $\alpha = 1,\ \beta > 1.$

If $f \in L^1(a,b)$, then M-L R-L lemma (II): $\lim\limits_{k\to\infty} \int_a^b E_{\alpha,\beta}\left(i^\alpha kx\right) f(x)dx = 0.$

If $f \in C^1([a,b])$, then

- for $0 < \alpha < 2$ and $\beta \geq \alpha + 1$, we have
  $\int_a^b E_{\alpha,\beta}\left(i^\alpha kx\right) f(x)dx = \mathcal{O}\left(\frac{\log(1+k)}{k}\right);$
- for $0 < \alpha < 2$ and $1 < \beta < \alpha + 1$, we have
  $\int_a^b E_{\alpha,\beta}\left(i^\alpha kx\right) f(x)dx = \mathcal{O}\left(k^{\frac{1-\beta}{\alpha}}\right);$
- for $0 < \alpha < 2$ and $\beta = \alpha$, we have $\int_a^b E_{\alpha,\alpha}\left(i^\alpha kx\right) f(x)dx = \mathcal{O}\left(k^{-1}\right).$

Thank you

ANALYSIS PDE UGENT

WORKSHOP FRACTIONAL CALCULUS

Think fractional gain integral