Determination of the order of fractional derivative for subdiffusion equations

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[1] R. Ashurov, S. Umarov. *Determination of the order of fractional derivative for subdiffusion equations.* arXiv:submit/3190665[math-ph]22 May 2020.

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[2] R. Ashurov, Y. Fayziev. *Uniqueness and existence for inverse problem of determining an order of time-fractional derivative of subdiffusion equation.* arXiv:submit/3213311[math.AP]6 Jun 2020

[3] R. Ashurov. *Inverse problems of determining an order of time-fractional derivative for a wave equation.* arXiv:submit/3213310[math.AP]6 Jun 2020

[4] R. Ashurov, Y. Fayziev. *Determination of fractional order and source term in a fractional subdiffusion equation.* (In Russian.)

[5] Sh. Alimov, R. Ashurov. *Inverse problems of determining an order of the Caputo time-fractional derivative for a subdiffusion equation.* arXiv:submit/3214612[math.AP]7 Jun 2020

Let $0 < \rho < 1$. Consider

$$\partial_t^\rho u(x,t) = \Delta u(x,t), \quad x \in \Omega \subset R^N,\ t > 0, \ (1)$$

$$Bu(x,t) \equiv \frac{\partial u(x,t)}{\partial n} = 0, \quad x \in \partial\Omega,\ t \geq 0, \quad (2)$$

$$\lim_{t \to 0} \partial_t^{\rho-1} u(x,t) = \varphi(x), \quad x \in \overline{\Omega}, \qquad (3)$$

where n is the unit outward normal vector of $\partial\Omega$. This is Direct or Forward problem.

Consider the spectral problem

$$-\triangle v(x) = \lambda v(x), \quad x \in \Omega;$$

$$Bv(x) = 0, \quad x \in \partial\Omega.$$

This problem has a complete in $L_2(\Omega)$ set of orthonormal eigenfunctions $\{v_k(x)\}$ and a countable set of nonnegative eigenvalues $\{\lambda_k\}$. Note $\lambda_1 = 0$, $v_1(x) = |\Omega|^{-1/2}$.

O.A. Ladyjinskaya, *Mixed problem for a hyperbolic equation.* Gostexizdat (1953).

Fractional integration in the Riemann - Liouville sense of order $\rho < 0$:

$$\partial_t^\rho f(t) = \frac{1}{\Gamma(-\rho)} \int_0^t \frac{f(\xi)}{(t-\xi)^{\rho+1}} d\xi, \quad t > 0,$$

The Riemann - Liouville fractional derivative of order ρ, $k - 1 < \rho \leq k$:

$$\partial_t^\rho f(t) = \frac{d^k}{dt^k} \partial_t^{\rho-k} f(t).$$

The Caputo:

$$D_t^\rho f(t) = \partial_t^{\rho-k} \frac{d^k}{dt^k} f(t).$$

Under certain conditions on initial function φ the solution of (1)-(3) exists and is unique: $u(x,t;\rho)$.

The purpose of this paper is not only to find the solution $u(x,t)$, but also to determine the order $\rho \in (0,1)$ of the time derivative.

To do this one needs an extra condition.

Using the classical Fourier method, we prove that the only condition

$$f(\rho; t_0) \equiv \int_{\Omega} u(x, t_0) dx = d_0, \qquad (4)$$

where $t_0 \geq 1$ is an observation time, recovers the order $\rho \in (0, 1)$, and if $\{u_1(x, t), \rho_1\}$ and $\{u_2(x, t), \rho_2\}$, then $u_1(x, t) \equiv u_2(x, t)$ and $\rho_1 = \rho_2$.

The quantity $f(\rho; t_0) \Rightarrow$ the projection of the solution $u(x, t_0)$ onto the first eigenfunction. Remember: $v_1(x) = |\Omega|^{-1/2}$.

Abdallah El Hamidi and Ali Tfayli. *Identification of the derivative order in fractional differential equations*. Received: 18 July 2019 DOI: 10.1002/mma.6175.

The author showed, that $\frac{\partial u(\rho)}{\partial \rho}$ satisfies the same problem as $u(\rho)$ does, but with other source terms.

PhD student of prof. Mokhtar Kirane

Problem (1)-(3), (4) is an important type of **inverse problems**, namely to determining of the order of fractional derivative in a subdiffusion equation. Usually the extra conditions has the form

$$u(x_0, t) = d(t),\ 0 < t < T,$$

at a monitoring point $x_0 \in \overline{\Omega}$.

The uniqueness (note, this is very important from application point of view) for this inverse problem have been studied by a number of authors.

[1] J. Cheng, J. Nakagawa, M. Yamamoto, T. Yamazaki, Uniqueness in an inverse problem for a one-dimensional fractional diffusion equation. *Inverse Prob.* **4** (2009), 1–25.

The first mathematical result for the coefficient inverse problem for a fractional differential equation.

[2] S. Tatar, S. Ulusoy, A uniqueness result for an inverse problem in a **space-time** fractional diffusion equation. *Electron. J. Differ. Equ.*, **257** (2013), 1–9.

[3] Z. Li, M. Yamamoto, Uniqueness for inverse problems of determining orders of **multi-term** time-fractional derivatives of diffusion equation. *Appl. Anal.*, **94** (2015), 570–579.

[4] Z. Li, Y. Luchko, M. Yamamoto, Analyticity of solutions to a **distributed order** time-fractional diffusion equation and its application to an inverse problem. *Comput. Math. Appl.*, **73** (2017), 1041–1052.

[5] X. Zheng, J. Cheng, H. Wang, Uniqueness of determining the variable fractional order in **variable-order** time-fractional diffusion equations. *Inverse problems.* **35** (2019), 1–11.

The main result of this work is based on Lemma 4.1.

But in our opinion, this lemma is questionable. We constructed a counterexample and sent to the authors. (There exist functions whose Fourier series converge to zero in a certain region, but not all Fourier coefficients are zero).

[6] Y. Hatano, J. Nakagawa, S. Wang, M. Yamamoto, Determination of order in fractional diffusion equation, *J. Math-for-Ind.*, **5A** (2013), 51–57.

If $u(x,0) = \varphi \in C_0^\infty(\Omega)$ and $\Delta\varphi(x_0) \neq 0$, then

$$\rho = \lim_{t \to 0} \left[t\partial_t u(x_0, t)[u(x_0, t) - \varphi(x_0)]^{-1} \right]$$

Prof. Karimov sent: Mirko D'Ovidio, Paola Loreti, Alireza Momenzadeh, Sima Sarv Ahrabi *Determination of order in linear fractional differential equations* Fract. Calc. Appl. Anal., Vol. 21, No 4 (2018), pp. 937948.

A survey paper: [7] Z. Li, Y. Liu, M. Yamamoto, Inverse problems of determining parameters of the fractional partial differential equations. *Handbook of fractional calculus with applications.* **2**, DeGruyter (2019). pp. 431- 442.

The paper

[8] Janno, J. *Determination of the order of fractional derivative and a kernel in an inverse problem for a generalized time-fractional diffusion equation*. Electronic J. Differential Equations V. 216(2016), pp. 1-28.

deals with the existence problem. The author considered a time-fractional diffusion equation with Caputo derivatives of order $0 < \rho < 1$.

Giving an extra boundary condition $Bu(\cdot,t) = h(t), 0 < t < T$ the author succeeded to prove the existence theorem for determining **the order** of the derivative and **the kernel** of the integral operator in the equation.

$\Omega = (x_1, x_2)$, and $x_0 \in [x_1, x_2]$. Additional information:

$$1)\, u(x_0, t) = d(t),\ t \in (0, T)$$

$$2)\, u_x(x_0, t) + a \cdot u(x_0, t) = d(t),\ t \in (0, T)$$

$$3)\ \int_{x_1}^{x_2} k(x) u(x, t) dx = d(t),\ t \in (0, T).$$

Our result shows, if we take $k(x) = v_1(x)$, only $d(t_0)$ recovers the order ρ.

Our result gives a positive answer to the question posed in review article

[7] Li, Z., Liu, Y., Yamamoto, M. *Inverse problems of determining parameters of the fractional partial differential equations*, Handbook of fractional calculus with applications. V. 2. DeGruyter. 2019. pp. 431- 442.

"It would be interesting to investigate inverse problem **by the value of the solution at a fixed time** as the observation data" (Conclusions and Open Problems section).

We pass to a rigorous statement of the result.

Definition. *A pair* $\{u(x,t), \rho\}$ *of the function* $u(x,t)$ *and the parameter* ρ *with the properties*

$$\rho \in (0,1),$$

$$\partial_t^\rho u(x,t), \triangle u(x,t) \in C(\bar{\Omega} \times (0,\infty)),$$

$$\partial_t^{\rho-1} u(x,t) \in C(\bar{\Omega} \times [0,\infty))$$

is called **the solution** *of inverse problem (1) - (3), (4).*

Function $u(x,t)$ *with these properties is called* **the solution** *of forward problem (1) - (3).*

Let the initial function satisfy the following conditions:

$$\varphi(x) \in C^{\lfloor \frac{N}{2} \rfloor}(\Omega), \tag{5}$$

$$D^{\alpha}\varphi(x) \in L_2(\Omega), \quad |\alpha| = \left\lfloor \frac{N}{2} \right\rfloor + 1, \tag{6}$$

and on the boundary $x \in \partial\Omega$

$$B\varphi(x) = B\triangle\varphi(x) = \cdots = B\triangle^{\left\lfloor \frac{N}{4} \right\rfloor}\varphi(x) = 0. \tag{7}$$

Theorem. *(Direct Problem). Let conditions (5) - (7) be satisfied. Then there exists a unique solution of the forward problem (1) - (3) and it has the representation*

$$u(x,t) = \sum_{j=1}^{\infty} \varphi_j t^{\rho-1} E_{\rho,\rho}(-\lambda_j t^{\rho}) v_j(x), \qquad (8)$$

which absolutely and uniformly converges on $x \in \bar{\Omega}$ *for each* $t \in (0, T]$.

Theorem. *(Inverse Problem). Let function* $\varphi(x)$ *satisfy the conditions (5) - (7) and* $\varphi_1 \neq 0$*. Then inverse problem (1) - (3), (4) has a unique solution* $\{u(x,t),\rho\}$ *if and only if*

$$0 < \frac{d_0}{\varphi_1} < 1. \qquad (9)$$

Remember:

$$f(\rho; t_0) \equiv \int_\Omega u(x,t_0)dx = d_0.$$

Proof of Theorem (Direct problem).

$$u(x,t) = \sum_{j=1}^{\infty} T_j(t) v_j(x), \quad t > 0,\ x \in \Omega,$$

for $T_j(t)$ we have

$$\partial_t^\rho T_j + \lambda_j T_j = 0, \quad \lim_{t\to 0} \partial_t^{\rho-1} T_j(t) = \varphi_j.$$

It is known:

$$T_j(t) = \varphi_j t^{\rho-1} E_{\rho,\rho}(-\lambda_j t^\rho),$$

where $E_{\rho,\mu}$ is the Mittag-Leffler function

$$E_{\rho,\mu}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\rho k + \mu)}.$$

Lemma. *Let* $\sigma > 1 + \frac{N}{4}$*. Then for any multi-index* α *satisfying* $|\alpha| \leq 2$ *the operator* $D^{\alpha}(\hat{A} + I)^{-\sigma}$ *(completely) continuously maps the space* $L_2(\Omega)$ *into* $C(\bar{\Omega})$*, and moreover, the following estimate holds*

$$\|D^{\alpha}(\hat{A} + I)^{-\sigma} g\|_{C(\Omega)} \leq C\|g\|_{L_2(\Omega)}.$$

Krasnoselski, M.A., Zabreyko, P.P., Pustilnik, E.I., Sobolevski, P.S. *Integral operators in the spaces of integrable functions* (in Russian), M. NAUKA (1966).

(Based on the Coercivity inequality.)

Suppose that for some $\tau > \frac{N}{4}$:

$$\sum_1^\infty (\lambda_j + 1)^{2\tau} |\varphi_j|^2 \leq C_\varphi < \infty. \qquad (10)$$

Consider the sum

$$S_k(x,t) = \sum_{j=1}^k v_j(x) \varphi_j t^{\rho-1} E_{\rho,\rho}(-\lambda_j t^\rho).$$

Since $(\hat{A} + I)^{-\tau-1} v_j(x) = (\lambda_j + 1)^{-\tau-1} v_j(x)$,

$$S_k(x,t) =$$

$$(\hat{A}+I)^{-\tau-1} \sum_{j=1}^k v_j(x)(\lambda_j+1)^{\tau+1} \varphi_j t^{\rho-1} E_{\rho,\rho}(-\lambda_j t^\rho).$$

Therefore, by virtue of Lemma , one has

$$||D^\alpha S_k^1||_{C(\Omega)} = ||D^\alpha(\widehat{A}+I)^{-\tau-1} \times$$

$$\times \sum_{j=1}^{k} v_j(x)(\lambda_j+1)^{\tau+1}\varphi_j t^{\rho-1}E_{\rho,\rho}(-\lambda_j t^\rho)||_{C(\Omega)}$$

$$\leq C|| \sum_{j=1}^{k} v_j(x)(\lambda_j+1)^{\tau+1}\varphi_j t^{\rho-1}E_{\rho,\rho}(-\lambda_j t^\rho)||_{L_2(\Omega)}.$$

Since $\tau > \frac{N}{4}$, then $\sigma = 1+\tau > 1+\frac{N}{4}$.

Using the orthonormality of the system $\{v_j\}$, we have

$$\leq C \sum_{j=1}^{k} \left|(\lambda_j + 1)^{\tau+1}\varphi_j t^{\rho-1} E_{\rho,\rho}(-\lambda_j t^{\rho})\right|^2.$$

For the Mittag-Leffler function with a negative argument we have an estimate

$$|E_{\rho,\rho}(-t)| \leq \frac{C}{1+t}, \quad t > 0.$$

$$\sum_{j=1}^{k}\left|(\lambda_j+1)^{\tau+1}\varphi_j t^{\rho-1}E_{\rho,\rho}(-\lambda_j t^{\rho})\right|^2$$

$$=\sum_{\lambda_j<t^{-\rho}}\left|(\lambda_j+1)^{\tau+1}\varphi_j t^{\rho-1}E_{\rho,\rho}(-\lambda_j t^{\rho})\right|^2$$

$$+\sum_{\lambda_j>t^{-\rho}}\left|(\lambda_j+1)^{\tau+1}\varphi_j t^{\rho-1}E_{\rho,\rho}(-\lambda_j t^{\rho})\right|^2$$

$$\leq Ct^{-2}(1+t^{\rho})^2\sum_{j=1}^{k}(\lambda_j+1)^{2\tau}|\varphi_j|^2$$

$$\leq Ct^{-2}(1+T^{\rho})^2C_{\varphi}.$$

V.A. Il'in, On the solvability of mixed problems for hyperbolic and parabolic equations. *Russian Math. Surveys,* 1960.:

$$\varphi(x) \in C^{\lfloor \frac{N}{2} \rfloor}(\Omega),$$

$$D^{\alpha}\varphi(x) \in L_2(\Omega), \quad |\alpha| = \left\lfloor \frac{N}{2} \right\rfloor + 1,$$

and on the boundary $x \in \partial\Omega$

$$B\varphi(x) = B\triangle\varphi(x) = \cdots = B\triangle^{\left\lfloor \frac{N}{4} \right\rfloor}\varphi(x) = 0.$$

then for some $\tau > \frac{N}{4}$:

$$\sum_{1}^{\infty}(\lambda_j + 1)^{2\tau}|\varphi_j|^2 \leq C_{\varphi} < \infty.$$

Proof of Theorem (Inverse problem).

Lemma. *If* $\varphi_1 \neq 0$*, then* $f(\rho; t_0)$ *as a function of* $\rho \in (0,1)$ *is strictly monotone. Moreover*

$$\lim_{\rho \to +0} f(\rho; t_0) = 0, \quad f(1; t_0) = \varphi_1. \qquad (11)$$

Remember:

$$f(\rho; t_0) \equiv \int_\Omega u(x, t_0) dx = d_0.$$

Remember:

$$u(x,t) = \sum_{j=1}^{\infty} \varphi_j t^{\rho-1} E_{\rho,\rho}(-\lambda_j t^\rho) v_j(x),$$

and $v_1(x) = |\Omega|^{-1/2}$; $f(\rho; t_0) \equiv \int_\Omega u(x, t_0) dx$.

$\{v_j(x)\}$ - orthonormal and $\lambda_1 = 0$, then

$$f(\rho; t_0) = \varphi_1 t_0^{\rho-1} E_{\rho,\rho}(0) = \frac{\varphi_1 t_0^{\rho-1}}{\Gamma(\rho)}. \qquad (12)$$

Let $\Psi(\rho)$ be the logarithmic derivative of the gamma function $\Gamma(\rho)$. Then $\Gamma'(\rho) = \Gamma(\rho)\Psi(\rho)$, and for $\rho \in (0, 1)$ we have $\Gamma(\rho) > 0$ and $\Psi(\rho) < 0$. Therefore,

$$\frac{d}{d\rho}\left(\frac{t_0^{\rho-1}}{\Gamma(\rho)}\right) = \frac{t_0^{\rho-1}}{\Gamma(\rho)}\Big[\ln t_0 - \Psi(\rho)\Big] > 0.$$

for $t_0 > 1$. Thus function $f(\rho; t_0)$ increases or decreases depending on sign of φ_1.

Bateman H. *Higher transcendental functions*, McGraw-Hill (1953).

Existence of ρ, which satisfies condition

$$f(\rho;t_0) = \int\limits_{\Omega} u(x,t_0)v_1(x)dx = d_0.$$

$$f(\rho;t_0) = \varphi_1 t_0^{\rho-1} E_{\rho,\rho}(0) = \frac{\varphi_1 t_0^{\rho-1}}{\Gamma(\rho)} = d_0. \quad (13)$$

Remember:

$$\lim_{\rho\to+0} f(\rho;t_0) = 0, \qquad f(1;t_0) = \varphi_1.$$

Therefore:

$$0 < \frac{\int\limits_{\Omega} u(x,t_0)dx}{\varphi_1} < 1.$$

[1] R. Ashurov, S. Umarov. *Determination of the order of fractional derivative for subdiffusion equations.* arXiv:submit/3190665[math-ph]22 May 2020.

$\partial_t^\rho u(x,t) + L(x,D)u(x,t) = 0$, $0 < \rho < 1$, $R - L$,

$\Omega \subset R^N$, Classical solution. $u(x,t) =?$, $\rho =?$.

Condition: $\lambda_1 = 0$, $d_0 = (u(x,t_0), v_1)$, $t_0 \geq 1$.

$$0 < \frac{d_0}{\varphi_1} < 1. \qquad (14)$$

[2] R. Ashurov, Y. Fayziev. *Uniqueness and existence for inverse problem of determining an order of time-fractional derivative of subdiffusion equation.* arXiv:submit/3213311[math.AP]6 Jun 2020

$\partial_t^\rho u(x,t) + A(x,D)u(x,t) = f(x,t)$, $0 < \rho < 1$, $R - L$,

$\Omega \subset R^N$, Classical solution. $u(x,t) =?$, $\rho =?$.

Condition: $\lambda_1 = 0$.

S. Agmon considered the following spectral problem

$$\begin{cases} A(x,D)v(x) = \lambda v(x), \quad x \in \Omega; \\ B_j v(x) = \sum\limits_{|\alpha| \le m_j} b_{\alpha,j}(x) D^{\alpha} v(x) = 0, \\ 0 \le m_j \le m-1,\ j = 1,2,...,l;\ x \in \partial\Omega. \end{cases} \tag{15}$$

condition $(A)) \Rightarrow \{v_k(x)\}_{k=1}^{\infty}$ and λ_k exist.

Agmon, S. *On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems*, Comm. Pure and Appl. Math. 15(1962), pp. 119-141.

$$\partial_t^\rho u(x,t) + A(x,D)u(x,t) = f(x,t), \tag{16}$$

$$B_j u(x,t) = 0,\ j = 1,2,...,l;\ x \in \partial\Omega, \tag{17}$$

$$\lim_{t\to 0} \partial_t^{\rho-1} u(x,t) = \varphi(x), \quad x \in \overline{\Omega}. \tag{18}$$

$$\int_\Omega u(x,t_0)v_1(x)dx = d_0, \quad t_0 \geq T_0, \tag{19}$$

$$T_0 = \begin{cases} 2, & \varphi_1 \cdot f_1 \geq 0, \\ 5 \cdot \max\left\{1, \dfrac{|\varphi_1|}{|f_1|}\right\}, & \varphi_1 \cdot f_1 < 0. \end{cases}$$

Theorem. *Let* $\tau > \frac{N}{2m}$ *and*

$$\varphi \in D(\hat{A}^\tau);$$

$$t^{1-\rho} f(x,t) \in D(\hat{A}^\tau),\, t \in [0,T]$$

$$F(t) = t^{1-\rho} || \hat{A}^\tau f(x,t) ||_{L_2(\Omega)} \in C[0,T].$$

Moreover, let $t_0 \geq T_0$ *be any fixed number and*

$$\lambda_1 = 0, \quad f_1 = constant, \quad \varphi_1^2 + f_1^2 \neq 0.$$

Then inverse problem (16)- (19) has a unique classical solution $\{u(x,t), \rho\}$ *if and only if*

$$\min\{f_1, \varphi_1 + t_0 f_1\} < d_0 < \max\{f_1, \varphi_1 + t_0 f_1\}.$$

[3] R. Ashurov. *Inverse problems of determining an order of time-fractional derivative for a wave equation.* arXiv:submit/3213310[math.AP]6 Jun 2020

$\partial_t^\rho u(t) + Au(t) = f(t)$, $1 < \rho < 2$, $R - L$,

$A : H \to H$ (selfadjoint; bounded or unbounded), Classical and generalized solutions. $u(t) =?$, $\rho =?$.

Condition $\lambda_1 = 0$.

Consider the problem:

$$\partial_t^\rho u(t) + Au(t) = f(t), \quad 0 < t \le T, \tag{20}$$

$$\lim_{t\to 0} \partial_t^{\rho-1} u(t) = \varphi, \quad \lim_{t\to 0} \partial_t^{\rho-2} u(t) = \psi, \tag{21}$$

$$(u(t_0), v_1) = d_0, \quad t_0 \ge T_0. \tag{22}$$

$$T_0 = \begin{cases} 2, & \varphi_1 \cdot \psi_1 \ge 0, \\ 2 \cdot \max\left\{1, \dfrac{|\psi_1|}{|\varphi_1|}\right\}, & \varphi_1 \cdot \psi_1 < 0. \end{cases}$$

Theorem. *Let* $t_0 \geq T_0$ *and*

$$\lambda_1 = 0, \quad f_1(t) \equiv 0, \quad \varphi_1^2 + \psi_1^2 \neq 0.$$

Then $\exists$ *unique* $\{u(x,t), \rho\}$ *if and only if*

$$\min\{\varphi_1, \varphi_1 t_0 + \psi_1\} < d_0 < \max\{\varphi_1, \varphi_1 t_0 + \psi_1\}.$$

This result can be interpreted as follows. The vibration of the string is usually perceived by us by the sound made by the string. The sound of a string is an overlay of simple tones corresponding to standing waves into which the vibration decomposes. The above result states: having heard only the first standing wave, one may uniquely determine the musical instrument, which is sending this wave.

[9] Zhiyuan Li and Zhidong Zhang. **Unique determination** *of fractional order and source term in a fractional diffusion equation from sparse boundary data*. arXiv:2003.10927v1[math.AP]24 Mar 2020.

$f(x) = \sum_{k=1}^{K} p_k(x)\chi_{t\in[c_{k-1},c_k)}$. $1/2 < \rho < 1 =$? and $f(x) =$?. Extra data: $\frac{\partial u}{\partial n}(x,t)$, $t \in (0,\infty)$, $x \in X_{ab} \subset \partial\Omega$.

”The order ρ can reflect some of the inhomogeneity of the medium, which with the source term usually can not be measured straightforwardly”.

[4] R. Ashurov, Y. Fayziev. *Determination of fractional order and source term in a fractional subdiffusion equation.*

$\partial_t^\rho u(x,t) + A(x,D)u(x,t) = f(x)$, $0 < \rho < 1$, $R - L$,

$\Omega \subset R^N$, Classical solution. $u(x,t) =?$, $f(x) =?$, $\rho =?$.

Condition: $\lambda_1 = 0$, $(u(x,t_0), v_1(x)) = d_0$,

$$\lim_{t\to 0} \partial_t^{\rho-1} u(x,t) = \varphi(x),\ u(x,T) = \psi(x).$$

Conditions:

$$\varphi_1^2 + \psi_1^2 \neq 0.$$

If $\varphi_1 \cdot \psi_1 \leq 0$, then $t_0 \in (1, T)$ and otherwise $t_0 \in (1, T)$ and

$$t_0 \in \begin{cases} \Big(1, \dfrac{\varphi_1}{\psi_1} \cdot T\Big) & \text{if} \quad \dfrac{\varphi_1}{\psi_1} \cdot T > 1, \\ \Big((2(\ln T + 1)\dfrac{\varphi_1}{\psi_1} \cdot T, T\Big) & \text{if} \quad \dfrac{\varphi_1}{\psi_1} \cdot T \leq 1. \end{cases}$$

Theorem. *Then* $\exists$ *unique* $\{u(x,t), f(x), \rho\}$ *if and only if*

$$\min\left\{\psi_1,\ \varphi_1\left[1-\frac{t_0}{T}\right]+\frac{t_0\psi_1}{T}\right\} < d_0 <$$

$$< \max\left\{\psi_1,\ \varphi_1\left[1-\frac{t_0}{T}\right]+\frac{t_0\psi_1}{T}\right\}.$$

[5] Sh. Alimov, R. Ashurov. *Inverse problems of determining an order of the Caputo time-fractional derivative for a subdiffusion equation.* arXiv:submit/3214612[math.AP]7 Jun 2020

$D_t^\rho u(t) + Au(t) = f(t)$, $0 < \rho < 1$, Caputo,

$A : H \to H$, Generalized solutions. $u(t) = ?$, $\rho = ?$.

No need of condition $\lambda_1 = 0$.

Nonhomogeneous boundary conditions.

$A : H \to H$ selfadjoint operator in H. By von Neumann's spectral theorem,

$$A = \int_{\mu}^{\infty} \lambda \, dP_\lambda, \quad \mu > 0.$$

where for a partition $\{P_\lambda\}$ of unity one has

$$\lim_{\lambda \to \infty} ||P_\lambda u - u|| = 0, \quad u \in H.$$

Consider the Cauchy type problem:

$$D_t^{\rho} u(t) + Au(t) = 0, \quad 0 < t \leq T, \tag{23}$$

$$u(0) = \varphi, \tag{24}$$

where φ is given vector in H.

Example 1. $A = -\Delta$ in $L_2(R^n)$. Spectrum is continuous.

Example 2. N-dimensional quadratic matrix $A = \{a_{i,j}\}_{i,j=1}^N$ and $H = R^N$. Problem (23), (24) $\Rightarrow$ the Cauchy problem for a linear system of fractional differential equations.

Example 3. A^{-1} is compact. In this case the spectrum is

$$\mu = \lambda_1 \leq \lambda_2 \leq ...$$

Physical examples, discussed in

Ruzhansky M., Tokmagambetov N., Torebek B.T. Inverse source problems for positive operators. I: Hypoelliptic diffusion and subdiffusion equations. // J. Inverse Ill-Possed Probl. 2019. V. 27. P. 891-911.

Example 3.1. Fractional Sturm-Liouville operators,

Example 3.2. Differential models with involution,

Example 3.3. Fractional Laplacians....

Extra data:

$$W(t_0,\rho) \equiv (Au(t_0), u(t_0)) = ||A^{\frac{1}{2}}u(t_0)||^2 = d_0. \tag{25}$$

Let $\mu > 1 + \ln 2$ and for any $t_0 \geq 2$, such that $t_0 < e^{\mu-1}$ set

$$\rho_0 = \left(\frac{1 + \ln t_o}{\mu}\right)^{1/3} < 1. \tag{26}$$

Lemma. *For any $\varphi \in H$ function $W(t_0,\rho)$ is monotonously decreasing with respect to $\rho \in [\rho_0, 1]$.*

Theorem. *Let the number* W^* *satisfy the condition*

$$W(t_0, 1) < W^* < W(t_0, \rho_0).$$

Then there exists the unique number $\rho^* \in [\rho_0, 1]$ *such that the solution* $u(t)$ *of problem (23), (24) with* $\rho = \rho^*$ *satisfies the equation*

$$||A^{\frac{1}{2}} u(t_0)||^2 = W^*.$$

Let $\Phi : R_+ \to R$, continuous function such that, if

$$\Phi(A)u(t) = \int_{\mu}^{\infty} \Phi(\lambda)\, dP_\lambda u(t),$$

then $D(\Phi(A)) \subseteq D(A)$. Extra data:

$$W(t_0, \rho) \equiv ||\Phi(A)u(t_0)||^2 = d_0, \qquad (27)$$

where t_0 is a fixed time instant.

Example 1. $\Phi(\lambda) \equiv 1$, then $W(t, \rho) = ||u(t)||^2$.

In the Conclusions and Open Problems section of

[7] Li, Z., Liu, Y., Yamamoto, M. *Inverse problems of determining parameters of the fractional partial differential equations*, Handbook of fractional calculus with applications. V. 2. DeGruyter. 2019. pp. 431- 442.

" The studies on inverse problems of the recovery of the fractional orders are far from satisfactory since all the publications either assumed **the homogeneous boundary condition**...."

Nonhomogeneous boundary condition:

$$\begin{cases} D_t^{\rho} u(x,t) - \triangle u(x,t) = 0, \quad x \in \Omega, \quad t > 0; \\ u(x,t) = c_0, \; x \in \partial\Omega, \quad t > 0; \\ u(x,0) = \psi(x), \quad x \in \overline{\Omega}, \end{cases} \tag{28}$$

where c_0 - constant, $\psi(x)$ is a given function.

Thank you for your attention!

[1] R. Ashurov, A. Muhiddinova. *Initial-boundary value problem for a time-fractional subdiffusion equation with an arbitrary elliptic differential operator.*

$\partial_t^\rho u(x,t) + A(x,D)u(x,t) = f(x,t)$, $0 < \rho < 1$, $R - L$,

$\Omega \subset R^N$, Classical solution. $u(x,t) =$?

[2] R. Ashurov, A. Muhiddinova. *Inverse problem for determining of a source function for subdiffusion equation.*

$\partial_t^\rho u(x,t) + A(x,D)u(x,t) = f(x)$, $0 < \rho < 1$, $R-L$,

$\Omega \subset R^N$, Classical solution. $u(x,t)$ =?, $f(x)$ =?.