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Bounded Solutions of Semilinear Fractional Schrödinger Differential and Difference Equations

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This is a discuss on the initial value problem for semilinear fractional Schrödinger integro-differential equation

$$i\frac{du}{dt} + Au = \int_0^t f\left(s, D_s^\alpha u(s)\right) ds, 0 < t < T, u\left(0\right) = 0$$

in a Hilbert space H with a self-adjoint positive definite operator A. First and second order of accuracy difference schemes for the approximate solution of differential problem are presented. Theorems on existence and uniqueness of the bounded solutions of these semilinear Schrödinger differential and difference problems are established. In practice, existence and uniqueness theorems for a bounded solution of the initial boundary value one-dimensional problem with nonlocal condition and multi-dimensional problem with local condition on the boundary are proved. Numerical results and explanatory illustrations are presented on one and multi-dimensional problems to show the validation of the theoretical results.

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2. A fractional Schrödinger integro-differential equation
3. A first order of accuracy difference scheme
4. R-modified Crank-Nicholson difference schemes
5. A second order of accuracy implicit difference scheme
6. Numerical results
7. Conclusion and our future plans

1. Introduction

In recent years, fractional calculus paves the way for scientists in various fields to model their nonlinear phenomena more precisely [1]-[5].

[1]. Podlubny I., Fractional Differential Equations, Academic Press, New York, 1999.

[2]. Lavoie, J. L, Osler, T. J., Tremblay, R., Review, Vol. 18 (2), pp. 240-268, 1976.

[3]. Diethelm, K., The analysis of fractional differential equations, Lecture Notes in Mathematics, Vol. 2004, Springer-Verlag, Berlin, 2010.

[4]. Ashyralyev, A., Hamad, A., FCAA, Vol.22(2), pp. 302-325, 2019.

[5]. Kiryakova, V., FCAA, Vol.17(4), pp. 977-1000, 2014.

This permits in new application fields for fractional equations: population dynamics, image processing, acoustics, electromagnetism, signal processing, information sciences, communications etc.[6]-[9].

[6]. Yan, Y., Ekaka-a, E.N., Journal of Franklin Institute, Vol. 348 (2011) 2744-2758.

[7]. Tarasov, V. E., Journal of Engineering Mechanics, Vol. 143 (5), Article ID: D4016001, 2017.

[8]. Baleanu, D., Inc, M., Aliyu, A.I., Yusuf, A., Superlattices and Microstructures, Vol. 111, pp. 546-555, 2017.

[9]. Bhrawy, A. H.; Abdelkawy, M. A., Journal of Computational Physics, Vol. 294, pp. 462-483, 2015.

However, modelling of real world problems in differential equation forms raise the necessity for exact or approximate solutions of these differential problems. It is well known that nonlinear problems are difficult to handle with either analytical or numerical approaches. Fractional Schrödinger problems appear in different studies with fractional order derivative in time, space or both directions. Nonlinear versions are also various and they are studied extensively in many different ways. Nonlinear fractional Schrödinger problems are studied in many papers including but not limited to [10]-[15].

[10].Yang, Z., International Journal of Computer Mathematics, Vol. 93 (3) pp.609-626, 2016.

[11]. Secchi, S., Journal of Mathematical Physics, Vol. 54 (3), Article ID: 031501, 2013.

[12]. Guo B., Han Y., Xin J., Applied Mathematics and Computation, Vol. 204 (1), pp. 468-477, 2008.

[13]. Zhang, J., Zhu, S., Journal of Dynamics and Differential Equations, Vol. 29 (3), pp. 1017-1030, 2017.

[14]. Ardila, A. H., Nonlinear Analysis, Volume 155, Pages 52-64, 2017

[15]. Kirane, M., Nabti, A., Z. Angew. Math. Phys. 66, 2015.

Furthermore, papers on physical properties of fractional Schrödinger equations prove the strong relation of mathematical improvements and the physical phenomena they refer [16]-[19].

[16]. Antoine, X., Tang, Q., Zhang, J., International Journal of Computer Mathematics, Vol. 95 (6-7), pp. 1423-1443, 2018.

[17]. Wang, P., Huang, C., Journal of Computational Physics, Vol. 293, pp. 238-251, 2015.

[18]. Ionescu, A. D.; Pusateri, F., Journal of Functional Analysis, Vol. 266 (1), pp. 139-176, 2014.

[19]. Li, D., Wang, J., Zhang, J., SIAM Journal on Scientific Computing, Vol. 39 (6), pp. A3067-A3088, 2017.

However, time fractional nonlinear problems are studied rarely in the literature [20]-[23].

[20]. Garrappa, R., Moret, I., Popolizio, M., Computers and Mathematics with Applications, Vol. 74(5), pp 977-992, 2017

[21]. Lashkarian, E., Hejazi, S. R., Mathematical Methods in the Applied Sciences, Vol. 41 (7) pp. 2664-2672, 2018.

[22]. Mohebbi, A., Abbaszadeh, M., Dehghan, M., Engineering Analysis with Boundary Elements, Vol. 37 (2), pp. 475-485, 2013.

[23]. Shivanian, E., Jafarabadi, A., Numerical Methods for Partial Differential Equations, Vol. 33 (4), pp. 1043-1069, 2017.

Implemented methods include meshless techniques, matrix functions, finite element method, Lie group analysis method and so on. However, finite difference schemes are not investigated well for nonlinear time fractional Schrödinger equations. In the papers on finite difference methods for linear fractional Schrödinger equations, fractional derivative appears in space variables [24]-[27].

[24]. Ran, M., Zhang, C., Communications in Nonlinear Science and Numerical Simulation, Vol. 41, pp. 64-83, 2016.

[25]. Yang, Z., International Journal of Computer Mathematics, Vol. 93 (3), pp. 609-626, 2016.

[26]. Wang, P., Huang, C., Journal of Computational Physics, Vol. 293, pp. 238-251, 2015.

[27]. Wang, D., Xiao, A., Yang, W., Journal of Computational Physics, Vol. 272, pp. 644-655, 2014.

It is known that classical numerical techniques like finite difference method preserve their importance due to their well-established theory and useful properties like stability [28]-[34].

[28]. Han, H., Jin, J., Wu, X., Computers and Mathematics with Applications Vol. 50, 1345-1362, 2005.

[29]. Ashyralyev, A., Sobolevskii, P. E., New Difference Schemes for Partial Differential Equations, Birkhäuser Verlag, Basel, Boston, Berlin, 2004.

[30]. Ashyralyev, A., Sirma, A., Computers and Mathematics with Applications, Vol. 55 (3), pp. 392-407, 2008.

[31]. Ashyralyev, A., Hicdurmaz, B., Kybernetes, Vol. 40 (5-6), pp. 736-750, 2011.

[32]. Ashyralyev, A., Hicdurmaz, B., International Journal of Computer Mathematics, Vol. 89(13-14), pp. 1927-1936, 2012.

[33]. Ashyralyev, A., Boundary Value Problems, Article ID: 31, pp. 1-18, 2013.

[34]. Ashyralyev, A., Emirov, N., Cakir, Z., EJDE, 2014, Article ID: 97.

Multidimensional problems arise in real world problems frequently. Paper [35] is one of rare studies on high order approximation of a nonlinear two-dimensional fractional Schrödinger problem.

[35]. Zhao, X., Sun, Z. Z., Hao, Z. P., SIAM Journal on Scientific Computing, Vol. 36 (6), pp. A2865-A2886, 2014.

In recent publications, stability of initial boundary value problems for linear fractional Schrödinger differential equations are studied with different approaches (See [37]-[39],[BH] and references therein). However, the nonlinearity turns problem (1) into a challenging one.

[36]. Yan, Y., Pal, K., Ford, N. J., BIT Numerical Mathematics, Vol. 54 (2), pp. 555-584, 2014.

[37]. Garrappa, R., Moret, I., Popolizio, M., Journal of Computational Physics, Vol. 293, pp. 115-134, 2015.

[38]. Hicdurmaz, B., Ashyralyev, A., Computers & Mathematics with Applications, Vol. 72(6), pp. 1703-1713, 2016.

[39]. Ashyralyev, A., Hicdurmaz, B., Applied and Computational Mathematics, Vol. 17 (1), pp. 10-21, 2018.

[BH]. Hicdurmaz,H., *Initial Boundary Value Problems for Fractional Schrödinger Differential Equations*,PhD Thesis, Gebze Institute of Technology, Turkey, 2015.

In the present paper, we consider a nonlinear time fractional Schrödinger integro-differential problem

$$i\frac{du}{dt}+Au=\int\limits_0^t f\left(s,D_s^\alpha u(s)\right)ds,\ 0<t<T,0<\alpha<1,u(0)=0 \tag{1}$$

in a Hilbert space H with a self-adjoint positive definite operator A. Here

$$D_t^\alpha=D_{0+}^\alpha$$

is the standard Riemann-Liouville's derivative of order $\alpha\in(0,1)$.

In this paper, problem (1) is investigated with both theoretical and numerical approaches. Besides first order accurate difference scheme, various second order of accuracy difference schemes are presented. Since high order algorithms have great advantages in real world applications, the present paper fills the gap by presenting second order accurate approximation for n-dimensional nonlinear fractional Schrödinger problems with applying of operator approach.

2. A fractional Schrödinger integro-differential equation

Let H be a Hilbert space, A be a positive definite self-adjoint operator. Throughout this paper, $\left\{e^{iAt},\ t\geq 0\right\}$ is the strongly continuous exponential operator-function. Now let us give the following estimate that will be used soon:

$$\left\|e^{iAt}\right\|_{H\to H}\leq 1\quad . \tag{2}$$

A function $u(t)$ is called a solution of problem (1) if the following conditions are satisfied:

(i) $u(t)$ is continuously differentiable and $\int\limits_0^t f\left(s,D_s^\alpha u(s)\right)ds\in H$ on the segment $[0,T]$.

(ii) The element $u(t)$ belongs to $D(A)$ for all $t\in[0,T]$, and the function $Au(t)$ is continuous on segment $[0,T]$.

(iii) $u(t)$ satisfies the fractional differential equation and initial condition (1).

The procedure of proving theorem on the existence and uniqueness of a bounded solution of problem (1) is based on reducing this problem to an integral equation

$$u(t)=-\int_0^t ie^{iA(t-y)}\int_0^y f\left(z,D_z^\alpha u(z)\right)dzdy \tag{3}$$

in H^α and the use of successive approximations. Here, H^α $(0<\alpha<1)$ is the Banach space consisting of all abstract functions $v(t)$ having a fractional derivative of order α, defined on $[0,T]$ with values in H for which the following norm is finite:

$$\| v \|_{H^\alpha}=\max_{0\leq t\leq T}\| D_{0+}^\alpha v(t) \|_H+\max_{0\leq t\leq T}\|v(t)\|_H .$$

Moreover, the Banach space $C^{(\alpha)}\left([0,T],H\right)$ $(0<\alpha<1)$ is the space obtained by completion of all smooth H-valued functions $v(t)$ on $[0,T]$ in the norm

$$\|v\|_{C^{(\alpha)}([0,T],H)}=\| v \|_{C([0,T],H)}+\sup_{0\leq t<t+\tau\leq T}\left\|\frac{v(t+\tau)-v(t)}{\tau^\alpha}\right\|_H .$$

Here, $C\left([0,T],H\right)$ stands for the Banach space of continuous functions $v(t)$ defined on $[0,T]$ with values in H equipped with the norm

$$\| v \|_{C([0,T],H)}=\max_{0\leq t\leq T}\|v(t)\|_H .$$

We have that

$$u'(t)=-i\int_0^t f\left(z,D_z^\alpha u(z)\right)dz+\int_0^t Ae^{iA(t-y)}\int_0^y f\left(z,D_z^\alpha u(z)\right)dzdy$$

$$=-i\int_0^t e^{iA(t-y)}f\left(y,D_y^\alpha u(y)\right)dy. \tag{4}$$

Then

$$D_t^\alpha u(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{u'(p)dp}{(t-p)^\alpha} \tag{5}$$

$$=-\frac{i}{\Gamma(1-\alpha)}\int_0^t\int_0^p e^{iA(p-y)}f\left(y,D_y^\alpha u(y)\right)dy\frac{dp}{(t-p)^\alpha}.$$

The recursive formula for the solution of problem (1) is

$$D_t^\alpha u_j(t)=-\frac{i}{\Gamma(1-\alpha)}\int_0^t\int_0^p e^{iA(p-y)}f\left(y,D_y^\alpha u_{j-1}(y)\right)dy\frac{dp}{(t-p)^\alpha},$$

$$j=1,2,..., \tag{6}$$

Theorem 2.1. Assume the following hypotheses hold:

1. The function $f : [0,T] \times H^{\alpha} \longrightarrow H$ is continuous, that is

$$\|f(t, D_t^{\alpha} u(t))\|_H \leq \bar{M}, \tag{7}$$

2. Lipschitz condition holds uniformly with respect to t

$$\|f(t, D_t^{\alpha} u) - f(t, D_t^{\alpha} v)\|_H \leq L \|D_t^{\alpha} u - D_t^{\alpha} v\|_H. \tag{8}$$

Here and in future , $L, \bar{M}$ are positive constants. Then there exists a unique bounded solution $u(t)$ to problem (1) in H^{α}.

Proof. According to the method of recursive approximation (6), we get

$$D_t^{\alpha} u(t) = D_t^{\alpha} u_0(t) + \sum_{j=0}^{\infty} \left(D_t^{\alpha} u_{j+1}(t) - D_t^{\alpha} u_j(t) \right), \tag{9}$$

where

$$D_t^{\alpha} u_0(t) = 0.$$

Applying formula (6) and estimates (2) and (7), we get

$$\|D_t^{\alpha} u_{n+1}(t) - D_t^{\alpha} u_n(t)\|_H \leq \frac{L^n \bar{M} t^{(n+1)(2-\alpha)}}{\Gamma((n+1)(2-\alpha)+1)}$$

and

$$\|D_t^{\alpha} u_{n+1}(t)\|_H \leq \frac{\bar{M} t^{(2-\alpha)}}{\Gamma(3-\alpha)} + \frac{L \bar{M} t^{2(2-\alpha)}}{\Gamma(5-2\alpha)} + \cdots + \frac{L^n \bar{M} t^{(n+1)(2-\alpha)}}{\Gamma((n+1)(2-\alpha)+1)}$$

for any $n, n \geq 1$. From that and formula (9) it follows that

$$\begin{aligned} \|D_t^{\alpha} u(t)\|_H &\leq \|D_t^{\alpha} u_0(t)\|_H + \sum_{m=0}^{\infty} \|D_t^{\alpha} u_{m+1}(t) - D_t^{\alpha} u_m(t)\|_H \\ &\leq \sum_{m=0}^{\infty} \frac{L^m \bar{M} t^{(n+1)(2-\alpha)}}{\Gamma((n+1)(2-\alpha)+1)} < \infty, 0 \leq t \leq T \end{aligned} \tag{10}$$

which proves the existence of a bounded solution $u(t)$ of problem (1) in H^{α}. The boundedness of $\frac{du(t)}{dt}$ and $Au(t)$ in $C([0,T], H)$ norm follow from equation (1), formula (4), the triangle inequality and estimates (7), (10). Theorem 2.1 is proved.

Now, we consider the applications of abstract result to one dimensional nonlocal and a multi dimensional local problems.

First, the mixed problem for semilinear fractional Schrödinger equation

$$\begin{cases} i\frac{\partial u}{\partial t} - (a(x)u_x(t,x))_x + \delta u(t,x) = \int\limits_0^t f(s, D_s^{\alpha} u(s,x))\, ds, \\ 0 < t < T, 0 < x < 1, \\ u(0,x) = 0, \ x \in [0,1], \\ u(t,0) = u(t,1), \ u_x(t,0) = u_x(t,1), \ 0 \leq t \leq T \end{cases} \tag{11}$$

is considered. Assume that $a(x) \geq 0$ $(x \in (0,1))$ is the smooth function and $a(0) = a(1)$ and all compatibility conditions hold.

We introduce the Hilbert space $L_2[0,1]$ of all square integrable functions defined on $[0,1]$. This allows us to reduce mixed problem (11) to initial value problem (1) in a Hilbert space H^α with a self-adjoint operator A^x generated by problem (11).

Theorem 2.2. Assume the following hypotheses hold:

1 The function $f : [0,T] \times C^{(\alpha)}([0,T], L_2[0,1]) \longrightarrow L_2[0,1]$ be continuous function, that is

$$\|f(t, D_t^\alpha u(t))\|_{L_2[0,1]} \leq \bar{M} \tag{12}$$

in $[0,T] \times C^{(\alpha)}([0,T], L_2[0,1])$

2 Lipschitz condition holds uniformly with respect to t

$$\|f(t, D_t^\alpha u) - f(t, D_t^\alpha v)\|_{L_2[0,1]} \leq L\|D_t^\alpha u - D_t^\alpha v\|_{L_2[0,1]}. \tag{13}$$

Then there exists a unique solution to problem (11) which is bounded in H^α.

The proof of Theorem 2.2 is based on the abstract Theorem 2.1 and symmetry properties of the operator A^x generated by problem (11).

Second, let Ω be the unit open cube in the m-dimensional Euclidean space $R^m : \Omega = \{x = (x_1, ..., x_m) : 0 < x_j < 1,\ 1 \leq j \leq m\}$ with boundary S, $\overline{\Omega} = \Omega \cup S$. In $[0,T] \times \Omega$. The mixed boundary value problem for the multidimensional fractional Schrödinger equation

$$\begin{cases} i\frac{\partial u}{\partial t} - \sum\limits_{r=1}^{m} (a_r(x) u_{x_r})_{x_r} + \delta u(t,x) = \int\limits_0^t f\left(s, D_s^\alpha u(s,x)\right) ds, \\ \\ 0 < t < T, x = (x_1, \cdot\cdot\cdot, x_m) \in \Omega, \\ \\ u(0,x) = 0,\ x \in \overline{\Omega}, \\ \\ u(t,x) = 0,\ x \in S \end{cases} \tag{14}$$

is considered. Assume that $a_r(x)$ $(x \in \Omega)$ is the smooth function and $a_r(x) \geq 0$ and all compatibility conditions hold.

We introduce the Hilbert space $L_2(\overline{\Omega})$ defined on $\overline{\Omega}$. This allows us to reduce mixed problem (14) to initial value problem (1) in a Hilbert space H^α with a self-adjoint operator A^x generated by problem (14).

Theorem 2.3. Assume the following hypotheses:

1 The function $f : [0,T] \times C^{(\alpha)}\left([0,T], L_2(\overline{\Omega})\right) \longrightarrow L_2(\overline{\Omega})$ be continuous function, that is

$$\|f(t, D_t^\alpha u(t))\|_{L_2(\overline{\Omega})} \leq \bar{M} \tag{15}$$

in $[0,T] \times C^{(\alpha)}\left([0,T], L_2(\overline{\Omega})\right)$,

2 Lipschitz condition holds uniformly with respect to t

$$\|f(t, D_t^\alpha u) - f(t, D_t^\alpha v)\|_{L_2(\overline{\Omega})} \leq L\|D_t^\alpha u - D_t^\alpha v\|_{L_2(\overline{\Omega})}. \tag{16}$$

Then there exists a unique solution to problem (14) which is bounded in H^{α}.

The proof of Theorem 2.3 is based on the abstract Theorem 2.1 and the symmetry properties of the operator A^{x} generated by problem (14) and the coercivity theorem for the solution of the elliptic differential problem in $L_2(\overline{\Omega})$.

3. A first order of accuracy difference scheme

For the approximate solution of (1), applying the set of grid points

$$[0,T]_{\tau}=\{t_k : t_k=k\tau,\ 0\leq k\leq N,\ N\tau=T\}, \tag{17}$$

we propose the first order of accuracy difference scheme

$$\begin{cases} i\frac{u_k-u_{k-1}}{\tau}+Au_k=\tau\sum\limits_{l=1}^{k}f(t_{l-1},D^{1,\alpha}u_{l-1}), \\ t_l=l\tau,\ 1\leq l\leq k\leq N,\ u_0=0. \end{cases} \tag{18}$$

In order to establish existence and uniqueness theorem of difference scheme (18) as a discrete analogy of Theorem 2.1, we introduce $H_{\tau}^{1,\alpha}, H_{\tau}^{2,\alpha}(0<\alpha<1)$ normed space of all abstract mesh functions $v^{\tau}=\{v_k\}_{k=0}^{N}$ having a fractional difference derivative of order α, defined on $[0,T]_{\tau}$ with values in H for which the following norms are finite uniformly with τ:

$$\| v^{\tau} \|_{H^{1,\alpha}}=\max_{0\leq l\leq N}\| D^{1,\alpha}v_l \|_{H}+\max_{0\leq l\leq N}\|v_l\|_{H}, \tag{19}$$

$$\| v^{\tau} \|_{H^{2,\alpha}}=\max_{0\leq l\leq N}\| D^{2,\alpha}v_l \|_{H}+\max_{0\leq l\leq N}\|v_l\|_{H}. \tag{20}$$

Moreover, the Banach space $C^{(\alpha)}([0,T]_{\tau},H)\,(0<\alpha<1)$ is the normed space of all H-valued mesh functions $v^{\tau}=\{v_k\}_{k=0}^{N}$ on $[0,T]_{\tau}$ with the norm

$$\|v^{\tau}\|_{C^{(\alpha)}([0,T]_{\tau},H)}=\| v^{\tau} \|_{C([0,T]_{\tau},H)}+\sup_{0\leq k<k+l\leq N}\left\|\frac{v_{k+l}-v_k}{(l\tau)^{\alpha}}\right\|_{H}. \tag{21}$$

Here, $C([0,T]_{\tau},H)$ stands for the Banach space of the mesh functions v^{τ} defined on $[0,T]_{\tau}$ with values in H,equipped with the norm

$$\| v^{\tau} \|_{C([0,T]_{\tau},H)}=\max_{0\leq l\leq N}\|v_l\|_{H}. \tag{22}$$

In order to establish existence and uniqueness theorems for bounded solution of (18), this difference scheme is reduced to the following equivalent nonlinear equation

$$u_k=-i\sum_{m=1}^{k}R^{k-m+1}\sum_{l=1}^{m}f(t_{l-1},D^{1,\alpha}u_{l-1})\tau^{2}, \tag{23}$$

where

$$R=(I-i\tau A)^{-1}. \tag{24}$$

We have that

$$\frac{u_k-u_{k-1}}{\tau}=-i\sum_{m=1}^{k}R^{k-m+1}\sum_{l=1}^{m}f(t_{l-1},D^{1,\alpha}u_{l-1})\tau$$

$$+i\sum_{m=1}^{k-1}R^{k-m}\sum_{l=1}^{m}f(t_{l-1},D^{1,\alpha}u_{l-1})\tau$$

$$=-i\sum_{m=1}^{k}R^{k-m+1}f(t_{m-1},D^{1,\alpha}u_{m-1})\tau. \tag{25}$$

Then, we get

$$D_{\tau}^{1,\alpha}u_k=\frac{1}{\Gamma(1-\alpha)}\sum_{m=1}^{k}\frac{\Gamma(k-m+1-\alpha)}{(k-m)!}\frac{u_m-u_{m-1}}{\tau^{\alpha}}$$

$$=-i\frac{1}{\Gamma(1-\alpha)}\sum_{m=1}^{k}\frac{\Gamma(k-m+1-\alpha)}{(k-m)!}\sum_{l=1}^{m}R^{m-l+1}f(t_{l-1},D^{1,\alpha}u_{l-1})\tau^{2-\alpha}. \tag{26}$$

The recursive formula for the solution of difference scheme (18) is

$$i\frac{u_k^{(j)}-u_{k-1}^{(j)}}{\tau}+Au_k^{(j)}=\tau\sum_{l=1}^{k}f(t_{l-1},D^{1,\alpha}u_{l-1}^{(j-1)}),$$

$$1\leq k\leq N, u_0^{(j)}=0, j=1,2,...,u_k^{(0)} \text{ is given.} \tag{27}$$

From (26) and (27) it follows

$$D_{\tau}^{1,\alpha}u_k^{(j)}=-i\frac{1}{\Gamma(1-\alpha)}\sum_{m=1}^{k}\frac{\Gamma(k-m+1-\alpha)}{(k-m)!}\sum_{l=1}^{m}R^{m-l+1}f(t_{l-1},D^{1,\alpha}u_{l-1}^{(j-1)})\tau^{2-\alpha},$$

$$j=1,2,...,u_k^{(0)} \text{ is given.} \tag{28}$$

Theorem 3.1. Let the assumptions of Theorem 2.1 be satisfied. Then, there exists a unique solution $u^{\tau}=\{u_k\}_{k=0}^{N}$ of difference scheme (18) which is bounded in $H_{\tau}^{1,\alpha}$ of uniformly with respect to τ.

Proof. According to the method of recursive approximation (28), we get

$$D_{\tau}^{1,\alpha}u_k=D_{\tau}^{1,\alpha}u_k^{(0)}+\sum_{j=0}^{\infty}\left(D_{\tau}^{1,\alpha}u_k^{(j+1)}-D_{\tau}^{1,\alpha}u_k^{(j)}\right), \tag{29}$$

where

$$D_{\tau}^{1,\alpha}u_k^{(0)}=0.$$

Applying formula (28), estimates (7) and

$$\|R\|_{H\to H}\leq 1, \tag{30}$$

we get

$$\|D_{\tau}^{\alpha}u_k^{(n+1)}-D_{\tau}^{\alpha}u_k^{(n)}\|_H\leq L^n\bar{M}\sum_{p=0}^{n+1}\begin{pmatrix} n+1\\ p\end{pmatrix}\tau^{(1-\alpha)(n+1-p)}\frac{(t_k)^{p(1-\alpha)+n+1}}{\Gamma(p(1-\alpha)+n+2)}.$$

and

$$\|D_\tau^\alpha u_k^{(n+1)}\|_H \le \bar{M}\left\{\tau^{1-\alpha} t_k + \frac{(t_k)^{2-\alpha}}{\Gamma(3-\alpha)}\right\} + L\bar{M}\left\{\tau\frac{t_k^2}{\Gamma(3)} + 2\tau^{1-\alpha}\frac{(t_k)^{3-\alpha}}{\Gamma(4-\alpha)} + \frac{(t_k)^3}{\Gamma(4)}\right\}$$

$$+\cdots+L^n\bar{M}\sum_{p=0}^{n+1}\begin{pmatrix} n+1 \\ p \end{pmatrix}\tau^{(1-\alpha)(n+1-p)}\frac{(t_k)^{p(1-\alpha)+n+1}}{\Gamma(p(1-\alpha)+n+2)}$$

for any $n \ge 1$. From that and formula (29) it follows that

$$\|D_\tau^\alpha u_k\|_H \le \left\|D_\tau^\alpha u_k^{(0)}\right\|_H + \sum_{m=0}^{\infty}\left\|D_\tau^\alpha u_k^{(j+1)} - D_\tau^\alpha u_k^{(j)}\right\|_H$$

$$\le \bar{M}\left\{\tau^{1-\alpha} t_k + \frac{(t_k)^{2-\alpha}}{\Gamma(3-\alpha)}\right\} + L\bar{M}\left\{\tau\frac{t_k^2}{\Gamma(3)} + 2\tau^{1-\alpha}\frac{(t_k)^{3-\alpha}}{\Gamma(4-\alpha)} + \frac{(t_k)^3}{\Gamma(4)}\right\}$$

$$+\cdots+L^n\bar{M}\sum_{p=0}^{n+1}\begin{pmatrix} n+1 \\ p \end{pmatrix}\tau^{(1-\alpha)(n+1-p)}\frac{(t_k)^{p(1-\alpha)+n+1}}{\Gamma(p(1-\alpha)+n+2)} + \cdots, 0 \le k \le N$$

which proves the existence of a solution $u^\tau = \{u_k\}_{k=0}^N$ of difference scheme (18) which is bounded in $H_\tau^{1,\alpha}$of uniformly with respect to τ. Theorem 3.1 is proved.

Now, we consider the applications of abstract result to one dimensional nonlocal and a multi dimensional local problems.

Note that a study of discretization, over time only, of the initial value problem also permits one to include general difference schemes in applications if the differential operators A in space variables are replaced by the difference operators A_h that act in the Hilbert spaces and are uniformly self-adjoint positive defined in h for $0 < h \le h_0$

First, mixed problem (11) for the fractional Schrödinger equation is considered. The discretization of problem (11) is provided in two steps. To the differential operator A generated by problem (11), we assign the difference operator A_h^x by the formula

$$A_h^x \varphi^h(x) = \{-(a(x)\varphi_{\bar{x}})_{x,r} + \delta\varphi^r\}_1^{M-1}, \tag{31}$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi^r\}_0^M$ satisfying the conditions $\varphi^0 = \varphi^M,\ \varphi^1 - \varphi^0 = \varphi^M - \varphi^{M-1}$. With the help of A_h^x, we arrive at the initial value problem

$$\begin{cases} \frac{du^h(t,x)}{dt} + A_h^x u^h(t,x) = \int\limits_0^t f^h(s,x,D_s^\alpha u^h(s,x))ds, \\ 0 < t < T,\ x \in [0,1]_h, \\ u^h(0,x) = 0,\ x \in [0,1]_h. \end{cases} \tag{32}$$

In the second step, we replace problem (32) by first order of accuracy difference scheme (18)

$$\begin{cases} i\frac{u_k - u_{k-1}}{\tau} + A_h^x u_k = \sum\limits_{l=1}^{k} f_{l-1}^h(x)\tau,\ x \in [0,1]_h, \\ f_k^h(x) = f(t_k, x_n, D_s^{1,\alpha} u_k),\ t_k = k\tau,\ 1 \le k \le N-1, \\ u_0^h(x) = 0,\ x \in [0,1]_h, \end{cases} \tag{33}$$

Theorem 3.3. Let the assumptions of Theorem 3.1 be satisfied. Then, there exists a unique solution $\left\{u_k^h\right\}_{k=0}^N$ of difference scheme (33) which is bounded in $H_\tau^{1,\alpha}$ uniformly with respect to τ and h.

The proof of Theorems 3.3 is based on the abstract Theorem 3.1 and symmetry properties of the difference operator A_h^x defined by formula (31).

Second, initial boundary value problem (14) for the m-dimensional Schrödinger equation is considered. To the differential operator A generated by problem (14), we assign the difference operator A_h^x by the formula

$$A_h^x u^h(x) = -\sum_{r=1}^{m} \left(a_r(x) u_{\overline{x}_r}^h\right)_{x_r, j_r} \tag{34}$$

acting in the space of grid functions $u^h(x)$, satisfying the conditions $u^h(x) = 0$ ($\forall\, x \in S_h$). It is known that A_h^x is a self-adjoint positive definite operator in $L_2(\overline{\Omega}_h)$. With the help of A_h^x, we arrive at the initial value problem

$$\begin{cases} \frac{du^h(t,x)}{dt} + A_h^x u^h(t,x) = \int\limits_0^t f^h(s,x,D_s^\alpha u^h(s,x))ds, \\ \\ 0 < t < T,\ x \in \overline{\Omega}_h, \\ \\ u^h(0,x) = 0,\ x \in \overline{\Omega}_h. \end{cases} \tag{35}$$

We replace problem (14) by the following difference scheme

$$\begin{cases} i\frac{u_k - u_{k-1}}{\tau} + A_h^x u_k = \sum\limits_{l=1}^{k} f_{l-1}^h(x)\tau,\ x \in \overline{\Omega}_h, \\ \\ f_k^h(x) = f(t_k, x_n, D_s^{1,\alpha} u_k),\ t_k = k\tau,\ 1 \le k \le N-1, \\ \\ u_0^h(x) = 0,\ x \in \overline{\Omega}_h. \end{cases} \tag{36}$$

Theorem 3.4. Let the assumptions of Theorem 3.1 be satisfied. Then, there exists a unique solution $\left\{u_k^h\right\}_{k=0}^N$ of difference scheme (36) which is bounded in $H_\tau^{1,\alpha}$ uniformly with respect to τ and h.

The proof of Theorem 3.4 is based on the abstract Theorem 3.1 and symmetry properties of the difference operator A_h^x defined by formula (34) and on the theorem on coercivity inequality for the solution of the elliptic problem in L_{2h}.

4. R-modified Crank-Nicholson difference schemes

For the approximate solution of (1), applying the same set of grid points $[0,T]_\tau$,we present the $r-$modified Crank-Nicholson difference schemes

$$\begin{cases} i\frac{u_k - u_{k-1}}{\tau} + Au_k = \tau\sum\limits_{l=1}^{k} F_{l-1}\left(D^{2,\alpha} u_{l-1}\right), 1 \le k \le r, \\ \\ i\frac{u_k - u_{k-1}}{\tau} + \frac{1}{2}Au_k + \frac{1}{2}Au_{k-1} = \tau\sum\limits_{l=1}^{k} F_{l-1}\left(D^{2,\alpha} u_{l-1}\right), \\ \\ t_k = k\tau,\ r+1 \le k \le N,\ u_0 = 0, \end{cases} \tag{37}$$

where

$$\sum_{l=1}^{k} F_{l-1}\left(D^{2,\alpha}u_{l-1}\right) = \begin{cases} \frac{1}{2}\left(f(\frac{\tau}{2},\frac{1}{2}\left(D_{\tau}^{2,\alpha}u_1 + D_{\tau}^{2,\alpha}u_0\right)) + f(0, D_{\tau}^{2,\alpha}u_0)\right), k=1, \\ \sum\limits_{l=1}^{k-1} f(t_l, D_{\tau}^{2,\alpha}u_l) + \frac{1}{2}f(t_{k-\frac{\tau}{2}}, \frac{1}{2}\left(D_{\tau}^{2,\alpha}u_k) + D_{\tau}^{2,\alpha}u_{k-1}\right)) \\ +\frac{1}{2}f(0, D_{\tau}^{2,\alpha}u_0), 2 \le k \le N. \end{cases} \tag{38}$$

Here, second order of accuracy approximation formula for fractional derivative is defined by formula [AEC].

$$D_{\tau}^{2,\alpha}u_k = \begin{cases} \dfrac{1}{\Gamma(3-\alpha)2^{1-\alpha}\tau^{\alpha}}(-u_0+u_1),\ k=1, \\ \dfrac{3^{2-\alpha}}{2^{4-\alpha}\tau^{\alpha}\Gamma(4-\alpha)}\left\{(14\alpha-15)u_0 - (\alpha+33)u_1\right. \\ \left. +(2\alpha+3)u_2\right\},\ k=2, \\ \sum\limits_{m=2}^{k-1}\left\{\dfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\eta\left(k-m\right)\left(u_{m-1}-u_{m-2}\right)\right. \\ +\dfrac{(u_m-2u_{m-1}+u_{m-2})\tau^{-\alpha}}{\Gamma(1-\alpha)}\left(\dfrac{(k-m+1)\eta\left(k-m\right)}{(1-\alpha)}\right) \\ \left.+\dfrac{(u_m-2u_{m-1}+u_{m-2})\tau^{-\alpha}}{\Gamma(1-\alpha)}\left(\dfrac{\zeta(k-m)}{(2-\alpha)}\right)\right\} \\ +\dfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)2^{1-\alpha}}(u_{k-1}-u_{k-2}) \\ +\dfrac{\tau^{-\alpha}(3-\alpha)}{\Gamma(3-\alpha)2^{2-\alpha}}(u_k-2u_{k-1}+u_{k-2}),\ 3\le k\le N, \end{cases} \tag{39}$$

where

$$\eta(r) = (r+1/2)^{1-\alpha} - (r-1/2)^{1-\alpha}, \tag{40}$$
$$\zeta(r) = (r-1/2)^{2-\alpha} - (r+1/2)^{2-\alpha}. \tag{41}$$

[AEC]. Ashyralyev, A., Emirov, N., Cakir, Z., EJDE, 2014, Article ID: 97.

In order to establish existence and uniqueness theorem for bounded solution of (37), this difference scheme is reduced to the following equivalent nonlinear form

$$u_k = \begin{cases} -i\sum\limits_{j=1}^{k} R^{k-j+1}\sum\limits_{l=1}^{j} F_{l-1}\left(D^{2,\alpha}u_{l-1}\right)\tau^2, 1 \le k \le r, \\ \\ -i\sum\limits_{j=1}^{r} B^{k-r}R^{r-j+1}\sum\limits_{l=1}^{j} F_{l-1}\left(D^{2,\alpha}u_{l-1}\right)\tau^2 \\ \\ -i\sum\limits_{j=r+1}^{k} B^{k-j}C\sum\limits_{l=1}^{j} F_{l-1}\left(D^{2,\alpha}u_{l-1}\right)\tau^2, r+1 \le k \le N, \end{cases} \tag{42}$$

where

$$C = \left(I - i\frac{A\tau}{2}\right)^{-1}, B = \left(I + i\frac{A\tau}{2}\right)C, R = (I - iA\tau)^{-1}. \tag{43}$$

For easy computation, let's see the case when $r = 0$. We have that

$$u_k = -i\sum_{j=r+1}^{k} B^{k-j}C\sum_{l=1}^{j} F_{l-1}\left(D^{2,\alpha}u_{l-1}\right)\tau^2$$

and

$$\frac{u_k - u_{k-1}}{\tau} = -i\sum_{j=r+1}^{k} B^{k-j}C\sum_{l=1}^{j} F_{l-1}\left(D^{2,\alpha}u_{l-1}\right)\tau$$

$$+i\sum_{j=r+1}^{k-1} B^{k-1-j}C\sum_{l=1}^{j} F_{l-1}\left(D^{2,\alpha}u_{l-1}\right)\tau$$

$$= -i\sum_{l=1}^{k} B^{k-l}CF_{l-1}\left(D^{2,\alpha}u_{l-1}\right)\tau, 1 \le k \le N. \tag{44}$$

Then, we get

$$\frac{u_{k+1} - 2u_k + u_{k-1}}{\tau^2} = -iCF_0\left(D^{2,\alpha}u_0\right)$$

$$= -i\sum_{l=2}^{k} B^{k-l}C\left(F_{l-1}\left(D^{2,\alpha}u_{l-1}\right) - F_{l-2}\left(D^{2,\alpha}u_{l-2}\right)\right), 2 \le k \le N-2. \tag{45}$$

Applying formulas (39), (43) and (45), we get

$$D_\tau^{2,\alpha}u_0 = 0, D_\tau^{2,\alpha}u_1 = \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)2^{1-\alpha}}\left(-iCF_0\left(D^{2,\alpha}u_0\right)\right),$$

$$D_\tau^{2,\alpha}u_2 = -i\frac{3^{2-\alpha}\tau^{1-\alpha}}{2^{4-\alpha}\tau^\alpha\Gamma(4-\alpha)}C$$

$$\times\left((2\alpha+3)\left(\sum_{l=1}^{2} B^{2-l}F_{l-1}\left(D^{2,\alpha}u_{l-1}\right)\tau\right) + (\alpha-30)F_0\left(D^{2,\alpha}u_0\right)\tau\right),$$

$$D_\tau^{2,\alpha}u_k = \sum_{m=2}^{k-1}\left\{\frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\eta\left(k-m\right)\left(-i\sum_{l=1}^{m-1}B^{m-l-1}CF_{l-1}\left(D^{2,\alpha}u_{l-1}\right)\tau\right)\right.$$

$$+\frac{-i\tau^{1-\alpha}}{\Gamma(1-\alpha)}\left(\frac{(k-m+1)\eta\left(k-m\right)}{1-\alpha}+\frac{(k-m+1)\zeta\left(k-m\right)}{2-\alpha}\right)C$$

$$\times\left(\sum_{l=2}^{m}B^{m-l}\left(F_{l-1}\left(D^{2,\alpha}u_{l-1}\right)-F_{l-2}\left(D^{2,\alpha}u_{l-2}\right)\right)\tau+F_0\left(D^{2,\alpha}u_0\right)\right)$$

$$-\frac{i\tau^{1-\alpha}}{\Gamma(2-\alpha)2^{1-\alpha}}C\sum_{l=1}^{k-1}B^{k-1-l}F_{l-1}\left(D^{2,\alpha}u_{l-1}\right)\tau$$

$$+\frac{-i\tau^{1-\alpha}(3-\alpha)}{\Gamma(3-\alpha)2^{2-\alpha}}C$$

$$\times\left(F_0\left(D^{2,\alpha}u_0\right)+\sum_{l=2}^{k-1}B^{k-1-l}\left(F_{l-1}\left(D^{2,\alpha}u_{l-1}\right)-F_{l-2}\left(D^{2,\alpha}u_{l-2}\right)\right)\right), 3\le k\le N \tag{46}$$

The recursive formula for the solution of difference scheme (37) is

$$i\frac{u_k^{(j)}-u_{k-1}^{(j)}}{\tau}+Au_k^{(j)}=\tau\sum_{l=1}^{k}F_{l-1}\left(D^{2,\alpha}u_{l-1}^{(j)}\right), 1\le k\le r,$$

$$i\frac{u_k^{(j)}-u_{k-1}^{(j)}}{\tau}+\frac{1}{2}Au_k^{(j)}+\frac{1}{2}Au_{k-1}^{(j)}$$

$$=\sum_{l=1}^{k}F_{l-1}\left(D^{2,\alpha}u_{l-1}^{(j)}\right)\tau,$$

$$r+1\le k\le N, u_0^{(j)}=0, j=1,2,..., u_k^{(0)} \text{ is given.} \tag{47}$$

From (46) and (47) for $r=0$ it follows

$$D_\tau^{2,\alpha}u_0^{(j)}=0, D_\tau^{2,\alpha}u_1^{(j)}=\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)2^{1-\alpha}}\left(-iCF_0\left(D^{2,\alpha}u_0^{(j)}\right)\right),$$

$$D_\tau^{2,\alpha}u_2=-i\frac{3^{2-\alpha}\tau^{1-\alpha}}{2^{4-\alpha}\tau^\alpha\Gamma(4-\alpha)}C$$

$$\times\left((2\alpha+3)\left(\sum_{l=1}^{2}B^{2-l}F_{l-1}\left(D^{2,\alpha}u_{l-1}^{(j)}\right)\tau\right)+(\alpha-30)\,F_0\left(D^{2,\alpha}u_0^{(j)}\right)\tau\right),$$

$$D_\tau^{2,\alpha}u_k^{(j)} = \sum_{m=2}^{k-1}\left\{\frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\eta\left(k-m\right)\left(-i\sum_{l=1}^{m-1}CB^{m-l-1}F_{l-1}\left(D^{2,\alpha}u_{l-1}^{(j)}\right)\tau\right)\right.$$

$$+\frac{-i\tau^{1-\alpha}}{\Gamma(1-\alpha)}\left(\frac{(k-m+1)\eta\left(k-m\right)}{1-\alpha}+\frac{(k-m+1)\zeta\left(k-m\right)}{2-\alpha}\right)C$$

$$\times\left(\sum_{l=2}^{m}B^{m-l}\left(F_{l-1}\left(D^{2,\alpha}u_{l-1}^{(j)}\right)-F_{l-2}\left(D^{2,\alpha}u_{l-2}^{(j)}\right)\right)\tau+F_0\left(D^{2,\alpha}u_0^{(j)}\right)\right)$$

$$-\frac{i\tau^{1-\alpha}}{\Gamma(2-\alpha)2^{1-\alpha}}C\sum_{l=1}^{k-1}B^{k-1-l}F_{l-1}\left(D^{2,\alpha}u_{l-1}^{(j)}\right)\tau+\frac{-i\tau^{1-\alpha}(3-\alpha)}{\Gamma(3-\alpha)2^{2-\alpha}}C$$

$$\times\left(F_0\left(D^{2,\alpha}u_0^{(j)}\right)+\sum_{l=2}^{k-1}B^{k-1-l}\left(F_{l-1}\left(D^{2,\alpha}u_{l-1}^{(j)}\right)-F_{l-2}\left(D^{2,\alpha}u_{l-2}^{(j)}\right)\right)\right),3\leq k\leq N,$$

$$j=1,2,...,u_k^{(0)}\text{ is given.}$$

Theorem 4.1. Let the assumptions of Theorem 2.1 be satisfied. Then, there exists a unique solution $u^\tau=\{u_k\}_{k=0}^N$ of difference scheme (37) which is bounded in $H_\tau^{2,\alpha}$ uniformly with respect to τ.

Note that in a similar manner as section 3, we can construct r-modified Crank-Nicholson difference schemes for the approximate solutions of problems (11) and (14). Abstract Theorem 4.1 permit us to establish theorems on the existence of a bounded solution of these difference schemes uniformly with respect to τ and h.

5. A second order of accuracy implicit difference scheme

For the approximate solution of (1), applying the same set of grid points $[0,T]_\tau$,we present an implicit difference scheme of the the second order of accuracy

$$i\frac{u_k-u_{k-1}}{\tau}+A\left(I-\frac{i\tau A}{2}\right)u_k$$

$$=\left(I-\frac{i\tau A}{2}\right)\sum_{l=1}^{k}F_{l-1}\left(D^{2,\alpha}u_{l-1}\right)\tau,1\leq k\leq N,\ u_0=0, \tag{48}$$

where $\sum\limits_{l=1}^{k}F_{l-1}\left(D^{1,\alpha}u_{l-1}\right),1\leq k\leq N$ is defined by formulas (38) and (39).

In order to establish existence and uniqueness theorem for bounded solution of (48), this difference scheme is reduced to the following equivalent nonlinear equation

$$u_k=-i\sum_{m=1}^{k}\left(I-\frac{i\tau A}{2}\right)P^{k-m+1}\sum_{l=1}^{m}F_{l-1}\left(D^{2,\alpha}u_{l-1}\right)\tau^2,$$

where $P=\left(I-i\tau A-\frac{(\tau A)^2}{2}\right)^{-1}$. We have that

$$\frac{u_k-u_{k-1}}{\tau}=-i\sum_{m=1}^{k}\left(I-\frac{i\tau A}{2}\right)P^{k-m+1}\sum_{l=1}^{m}F_{l-1}\left(D^{2,\alpha}u_{l-1}\right)\tau$$

$$+i\sum_{m=1}^{k-1}\left(I-\frac{i\tau A}{2}\right)P^{k-m}\sum_{l=1}^{m}F_{l-1}\left(D^{2,\alpha}u_{l-1}\right)\tau$$

$$=-i\sum_{l=1}^{k}\left(I-\frac{i\tau A}{2}\right)P^{k-l+1}F_{l-1}\left(D^{2,\alpha}u_{l-1}\right)\tau,1\leq k\leq N. \tag{49}$$

Then, we get

$$\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}=-i\left(I-\frac{i\tau A}{2}\right)PF_0\left(D^{2,\alpha}u_0\right)$$

$$=-i\sum_{l=2}^{k}\left(I-\frac{i\tau A}{2}\right)P^{k-l+1}\left(F_{l-1}\left(D^{2,\alpha}u_{l-1}\right)-F_{l-2}\left(D^{2,\alpha}u_{l-2}\right)\right),2\leq k\leq N-2. \quad (50)$$

Applying formulas (39), (49) and (50), we get

$$D_\tau^{2,\alpha}u_0=0,D_\tau^{2,\alpha}u_1=\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)2^{1-\alpha}}\left(-i\left(I-\frac{i\tau A}{2}\right)PF_0\left(D^{2,\alpha}u_0\right)\right),$$

$$D_\tau^{2,\alpha}u_2=-i\frac{3^{2-\alpha}\tau^{1-\alpha}}{2^{4-\alpha}\tau^{\alpha}\Gamma(4-\alpha)}\left(I-\frac{i\tau A}{2}\right)P$$

$$\times\left((2\alpha+3)\left(\sum_{l=1}^{2}P^{2-l}F_{l-1}\left(D^{2,\alpha}u_{l-1}\right)\tau\right)+(\alpha-30)F_0\left(D^{2,\alpha}u_0\right)\tau\right),$$

$$D_\tau^{2,\alpha}u_k=\sum_{m=2}^{k-1}\left\{\frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\eta\left(k-m\right)\left(-i\sum_{l=1}^{m-1}\left(I-\frac{i\tau A}{2}\right)P^{m-l}F_{l-1}\left(D^{2,\alpha}u_{l-1}\right)\tau\right)\right.$$

$$+\frac{-i\tau^{1-\alpha}}{\Gamma(1-\alpha)}\left(\frac{(k-m+1)\eta\left(k-m\right)}{1-\alpha}+\frac{(k-m+1)\zeta\left(k-m\right)}{2-\alpha}\right)\left(I-\frac{i\tau A}{2}\right)P$$

$$\times\left(\sum_{l=2}^{m}P^{m-l}\left(F_{l-1}\left(D^{2,\alpha}u_{l-1}\right)-F_{l-2}\left(D^{2,\alpha}u_{l-2}\right)\right)\tau+F_0\left(D^{2,\alpha}u_0\right)\right)$$

$$-\frac{i\tau^{1-\alpha}}{\Gamma(2-\alpha)2^{1-\alpha}}\left(I-\frac{i\tau A}{2}\right)P\sum_{l=1}^{k-1}P^{k-1-l}F_{l-1}\left(D^{2,\alpha}u_{l-1}\right)\tau$$

$$+\frac{-i\tau^{1-\alpha}(3-\alpha)}{\Gamma(3-\alpha)2^{2-\alpha}}\left(I-\frac{i\tau A}{2}\right)P$$

$$\times\left(F_0\left(D^{2,\alpha}u_0\right)+\sum_{l=2}^{k-1}P^{k-1-l}\left(F_{l-1}\left(D^{2,\alpha}u_{l-1}\right)-F_{l-2}\left(D^{2,\alpha}u_{l-2}\right)\right)\right),3\leq k\leq N \quad (51)$$

The recursive formula for the solution of difference scheme (48) is

$$i\frac{u_k^{(j)}-u_{k-1}^{(j)}}{\tau}+A\left(I-\frac{i\tau A}{2}\right)u_k^{(j)}$$

$$=\left(I-\frac{i\tau A}{2}\right)\sum_{l=1}^{k}F_{l-1}\left(D^{2,\alpha}u_{l-1}^{(j)}\right)\tau,$$

$$1\leq k\leq N,u_0^{(j)}=0,j=1,2,...,u_k^{(0)}\text{ is given.} \quad (52)$$

From (51) and (52) it follows

$$D_\tau^{2,\alpha}u_0^{(j)}=0,D_\tau^{2,\alpha}u_1^{(j)}=\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)2^{1-\alpha}}\left(-i\left(I-\frac{i\tau A}{2}\right)PF_0\left(D^{2,\alpha}u_0^{(j)}\right)\right),$$

$$D_\tau^{2,\alpha}u_2 = -i\frac{3^{2-\alpha}\tau^{1-\alpha}}{2^{4-\alpha}\tau^\alpha\Gamma(4-\alpha)}\left(I - \frac{i\tau A}{2}\right)P$$

$$\times\left((2\alpha+3)\left(\sum_{l=1}^{2}P^{2-l}F_{l-1}\left(D^{2,\alpha}u_{l-1}^{(j)}\right)\tau\right) + (\alpha-30)\,F_0\left(D^{2,\alpha}u_0^{(j)}\right)\tau\right),$$

$$D_\tau^{2,\alpha}u_k^{(j)} = \sum_{m=2}^{k-1}\left\{\frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\eta\left(k-m\right)\left(-i\sum_{l=1}^{m-1}\left(I-\frac{i\tau A}{2}\right)P^{m-l}F_{l-1}\left(D^{2,\alpha}u_{l-1}^{(j)}\right)\tau\right)\right.$$

$$+\frac{-i\tau^{1-\alpha}}{\Gamma(1-\alpha)}\left(\frac{(k-m+1)\eta\left(k-m\right)}{1-\alpha}+\frac{(k-m+1)\zeta\left(k-m\right)}{2-\alpha}\right)\left(I-\frac{i\tau A}{2}\right)P$$

$$\times\left(\sum_{l=2}^{m}P^{m-l}\left(F_{l-1}\left(D^{2,\alpha}u_{l-1}^{(j)}\right)-F_{l-2}\left(D^{2,\alpha}u_{l-2}^{(j)}\right)\right)\tau + F_0\left(D^{2,\alpha}u_0^{(j)}\right)\right)$$

$$-\frac{i\tau^{1-\alpha}}{\Gamma(2-\alpha)2^{1-\alpha}}\left(I-\frac{i\tau A}{2}\right)P\sum_{l=1}^{k-1}P^{k-1-l}F_{l-1}\left(D^{2,\alpha}u_{l-1}^{(j)}\right)\tau$$

$$+\frac{-i\tau^{1-\alpha}(3-\alpha)}{\Gamma(3-\alpha)2^{2-\alpha}}\left(I-\frac{i\tau A}{2}\right)P$$

$$\times\left(F_0\left(D^{2,\alpha}u_0^{(j)}\right)+\sum_{l=2}^{k-1}P^{k-1-l}\left(F_{l-1}\left(D^{2,\alpha}u_{l-1}^{(j)}\right)-F_{l-2}\left(D^{2,\alpha}u_{l-2}^{(j)}\right)\right)\right), 3\le k\le N,$$

$$j = 1,2,...,u_k^{(0)} \text{ is given.} \tag{53}$$

Theorem 5.1. Let the assumptions of Theorem 2.1 be satisfied. Then, there exists a unique solution $u^\tau = \{u_k\}_{k=0}^N$ of difference scheme (48) which is bounded in $H_\tau^{2,\alpha}$ uniformly with respect to τ.

Note that in a similar manner as section 3, we can construct implicit second order of accuracy difference scheme for the approximate solutions of problems (11) and (14). Abstract Theorem 5.1 permit us to establish theorems on the existence of a bounded solution of these difference schemes uniformly with respect to τ and h.

6. Numerical results

Here, we implement iterated first and second order of accuracy difference schemes for one and multi-dimensional semilinear fractional Schrödinger problems.

First, we consider the following one-dimensional problem with the exact solution $u = (e^{it}-1)\sin x$.

$$\begin{cases} i\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} = \int\limits_0^t \sin\left(D^\alpha\left(u(s,x)-(e^{is}-1)\sin x\right)\right)ds \\ -\sin x, 0<x<\pi, 0<t<1, \\ u(0,x)=0, 0<x<\pi, \\ u(t,0)=u(t,\pi)=0, 0<t<1. \end{cases} \tag{54}$$

Applying difference schemes (18), (37) and (48) for the approximate solution of problem (54), we get the system of nonlinear equations. To get the solutions of the problem, we convert the problem into a system of matrices and we use fixed point iteration and a modified Gauss-elimination method to deal with the matrix equation. Throughout the experiments, iterations start with $u_n^k = 0$ and terminate when the error between each iteration becomes less than 10^{-7} in the given norm. The errors of the numerical solutions are computed by formula

$$E_M^N = \max_{1\leq k\leq N, 1\leq n\leq M-1} \left| u(t_k, x_n) - u_n^k \right|, \tag{55}$$

where $u(t_k, x_n)$ represents the exact solution and u_n^k represents the terminated numerical solution at (t_k, x_n) and the results are given in following tables.

Table 1: Errors of DS(18),(37) and (48)for α=0.25 when h=0.002π.

Method/N	10	20	40	80
DS (18)	4.80×10^{-2}	2.42×10^{-2}	1.22×10^{-2}	6.14×10^{-3}
DS (37), r=0	8.59×10^{-4}	2.08×10^{-4}	5.26×10^{-5}	1.54×10^{-5}
DS (37), r=1	8.36×10^{-3}	1.90×10^{-3}	4.37×10^{-4}	9.94×10^{-5}
DS (37), r=2	9.99×10^{-3}	2.53×10^{-3}	6.32×10^{-4}	1.56×10^{-4}
DS (48)	9.21×10^{-4}	2.31×10^{-4}	5.76×10^{-5}	1.43×10^{-5}

Table 2: Errors of DS (18),(37) and (48) for α=0.50 when h=0.002π.

Method/N	10	20	40	80
DS (18)	6.07×10^{-2}	3.07×10^{-2}	1.55×10^{-2}	7.77×10^{-3}
DS (37), r=0	9.04×10^{-4}	2.13×10^{-4}	5.29×10^{-5}	1.51×10^{-5}
DS (37), r=1	1.22×10^{-2}	2.76×10^{-3}	6.14×10^{-4}	1.35×10^{-4}
DS (37), r=2	1.04×10^{-2}	2.64×10^{-3}	6.60×10^{-4}	1.60×10^{-4}

Table 3: Errors of DS (18),(37) and (48) for α=0.75 when h=0.002π.

Method/N	10	20	40	80
DS (18)	6.10×10^{-2}	3.09×10^{-2}	1.55×10^{-2}	7.79×10^{-3}
DS (37), r=0	9.99×10^{-4}	2.24×10^{-4}	5.45×10^{-5}	1.55×10^{-5}
DS (37), r=1	2.02×10^{-2}	4.89×10^{-3}	1.12×10^{-3}	2.53×10^{-4}
DS (37), r=2	1.11×10^{-2}	2.86×10^{-3}	7.15×10^{-4}	1.73×10^{-4}

As it is seen in Table 1, Table 2 and Table 3 present the errors of difference schemes (18),(37) and (48) for $\alpha = 0.25, \alpha = 0.50$ and $\alpha = 0.75$, respectively. Some numerical results are given. If N are doubled and M is considered constant as $M = 250$, the value of errors decrease by a factor of approximately 1/2 for the first order of accuracy difference scheme (18) and the value of errors decrease by a factor of approximately 1/4 for the second order of accuracy difference schemes (37) and (48).

Second, we consider the following two-dimensional problem with the exact solution $u(t,x,y)=(e^{2it}-1)\sin x\sin y$.

$$\begin{cases} i\frac{\partial^{\alpha}u(t,x,y)}{\partial t}-\frac{\partial^{2}u(t,x,y)}{\partial x^{2}}-\frac{\partial^{2}u(t,x,y)}{\partial y^{2}}=\int\limits_{0}^{t}\sin(D^{\alpha}\left(u(s,x,y)-(e^{2is}-1)\sin x\sin y\right))ds \\ -2\sin x\sin y, 0<x,y<\pi, 0<t<1, \\ u(0,x,y)=0, 0\leq x,y\leq\pi, \\ u(t,0,y)=u(t,\pi,y)=u(t,x,0)=u(t,x,\pi)=0, 0\leq t\leq 1, 0\leq x,y\leq\pi, \end{cases} \quad (56)$$

Applying difference schemes (18), (37) and (48) for the approximate solution of problem

(56), we get the system of nonlinear equations. To get the solutions of the problem, we convert the problem into a system of matrices and we use fixed point iteration and a modified Gauss-elimination method to deal with the matrix equation. We get similary numerical resultsf or $\alpha=0.25, \alpha=0.50$ and $\alpha=0.75$, respectively. If N are doubled and M is considered constant as $M=100$, the value of errors decrease by a factor of approximately $1/2$ for the first order of accuracy difference scheme (18) and the value of errors decrease by a factor of approximately $1/4$ for the second order of accuracy difference schemes (37) and (48).

6. Conclusion and our future plans

1. In this study, the initial value problem (1) for semilinear fractional Schrödinger integro-differential equation in a Hilbert space with a self-adjoint positive definite operator.

- The main theorem on the existence and uniqueness of a bounded solution of problem (1) is established.
- The application of the main theorem to two semilinear fractional Schrödinger type partial differential equations is considered.
- The first and second order of accuracy stable difference schemes for the solution of problem (1) are presented.
- The theorem on the existence and uniqueness of a bounded solution uniformly with respect to time step of these difference schemes are established.
- The application of this theorem to two semilinear fractional Schrödinger type partial differential equations is considered.
- Numerical results are given for one and two dimensional fractional Schrödinger type partial differential equations.

Some parts of the present investigation are published in
[AABH] Ashyralyev, A., Hicdurmaz B., *Math Meth Appl Sci.* (2019), 1-21.

2. We are interested in studying the uniform two-step difference schemes and asymptotic formulas for the solution of initial value perturbation problem

$$\begin{cases} \varepsilon^2 u''(t) + iu'(t) + Au(t) = \int\limits_0^t f\left(s, D_s^\alpha u(s)\right) ds, \quad t > 0, \\ \\ u(0) = 0, \quad u'(0) = 0 \end{cases}$$

for a semilinear delay hyperbolic equation in a Hilbert space H with the self adjoint positive definite operator A and with $\varepsilon \in (0, \infty)$ parameter multiplying the highest order derivative term.

Earlier time in [AF] the uniform difference schemes and asymptotic formulas for the solution of initial value perturbation problem for a linear hyperbolic equation in a Hilbert space with the self adjoint positive definite operator and with $\varepsilon \in (0, \infty)$ parameter multiplying the highest order derivative term were presented and investigated.

[AF]. A. Ashyralyev and H.O. Fattorini, *SIAM J. Math. Anal.* **23**(1) (1992) 29-54.

3. Investigate the bounded solution of the identification problem for semilinear fractional Schrödinger integro-differential equation

$$iu'(t) + Au(t) = p + \int_0^t f\left(s, D_s^\alpha u(s)\right) ds, 0 < t < T, u\left(0\right) = 0, u\left(T\right) = \varphi$$

in a Hilbert space H with the self adjoint positive definite operator A.

[AM]. A. Ashyralyev and M. Urun, A second order of accuracy difference scheme for Schrödinger equations with an unknown parameter, *Filomat* 28(5) (2014) 981–993.

4. Investigate the bounded solution of the initial value problem for semilinear fractional Schrödinger integro-differential equation with time delay

$$\begin{cases} iu'(t) + Au(t) = \int\limits_0^t f\left(s, D_s^\alpha u(s-w)\right) ds, 0 < t < \infty, \\ u(t) = \varphi(t), -w \le t \le 0 \end{cases}$$

is considered in a Hilbert space H with a self-adjoint positive definite operator A.

[AA] A. Ashyralyev and D. Agirseven, On the stable difference schemes for the Schrödinger equation with time delay, *Computational Methods in Applied Mathematics* **20**(1) (2020) 27-38.

5. We are interested in studying the boundedness solution of the initial value problem for involutory semilinear fractional Schrödinger integro-differential equation

$$iu'(t) + Au(t) + bAu(-t) = \int_0^t f\left(s, D_s^\alpha u(s)\right) ds, -\infty < t < \infty, u\left(0\right) = 0.$$

[AATAAS]A. Ashyralyev, T. A. Hidayat and A. Sarsanbi, On the stability of Schrodinger type involutory differential equations. In *Functional Analysis in Interdisciplinary Applications* of ICAAM 2018. Springer Proceedings in Mathematics Statistics,2020, inpress.

Thanks for your attendance