Reconstruction of time-dependent sources and scalar parameters of fractional diffusion equations from final measurements

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Notation of fractional integral derivatives and Riemann-Liouville and Caputo fractional derivatives:

$$I^\beta w(t) = \int_0^t \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)} w(\tau) d\tau,$$

$$^R D^\beta w(t) = \frac{d}{dt} \int_0^t \frac{(t-\tau)^{-\beta}}{\Gamma(1-\beta)} w(\tau) d\tau,$$

$$^C D^\beta w(t) = \int_0^t \frac{(t-\tau)^{-\beta}}{\Gamma(1-\beta)} w'(\tau) d\tau,$$

where $0 < \beta < 1$.

Fractional diffusion equation in Riemann-Liouville form:

$$u_t(t,x) = {}^R D^{1-\beta}(\lambda \Delta u)(t,x) + Q(t,x),$$

where $\lambda > 0$, Δ is the Laplacian and Q is the source term.

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The same equation in Caputo form:

$${}^C D^{\beta} u(t,x) = \lambda \Delta u(t,x) + F(t,x), \quad \text{where} \quad F = I^{1-\beta} Q.$$

Let Q have the form

$$Q(t,x) = q(t)f(x).$$

Then

$$F(t,x) = I^{1-\beta}q(t)f(x).$$

Let $\Omega \subset \mathbb{R}^m$ be open bounded domain. Direct problem:

$$\left.\begin{aligned} &{}^C D^\beta u(t,x) = \lambda \Delta u(t,x) + I^{1-\beta} q(t) f(x), \ t \in (0,T),\ x \in \Omega, \\ &u(t,x) = 0, \ t \in (0,T),\ x \in \partial\Omega, \\ &u(0,x) = \varphi(x), \ x \in \Omega. \end{aligned}\right\} (1)$$

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Eigenvalues and eigenfunctions of $-\Delta$:

$$-\Delta v_k(x) = \mu_k v_k(x),\ x \in \Omega, \quad v_k(x) = 0,\ x \in \partial\Omega,$$

It holds $0 < \mu_1 \leq \mu_2 \leq \ldots$, $\mu_k \to \infty$
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Eigenvalues and eigenfunctions of $-\lambda\Delta$:

$$-\lambda\Delta v_k(x) = \lambda_k v_k(x),\ x \in \Omega, \quad v_k(x) = 0,\ x \in \partial\Omega.$$

It holds

$$\lambda_k = \lambda\mu_k.$$

Expansions of data and state function u:

$$f(x) = \sum_{k=1}^{\infty} f_k v_k(x),$$

$$\varphi(x) = \sum_{k=1}^{\infty} \varphi_k v_k(x),$$

$$\psi(x) = \sum_{k=1}^{\infty} \psi_k v_k(x),$$

$$u(t,x) = \sum_{k=1}^{\infty} u_k(t) v_k(x).$$

The Fourier coefficients of u are expressed as

$$u_k(t) = \varphi_k E_\beta(-\lambda_k t^\beta) + f_k \int_0^t (t-\tau)^{\beta-1} E_{\beta,\beta}(-\lambda_k (t-\tau)^\beta) I^{1-\beta} q(\tau) d\tau,$$

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into account we obtain the following basic relations:

$$\psi_k = \varphi_k E_\beta(-\lambda_k T^\beta) + f_k \int_0^T E_\beta(-\lambda_k (T-\tau)^\beta) q(\tau) d\tau, \quad k \in \mathbb{N}. \quad (3)$$

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Assume that there exists a subsequence k_i, $i \in \mathbb{N}$, such that

$$f_{k_i} \neq 0, \quad i \in \mathbb{N}.$$

Then

$$\int_0^T E_\beta(-\lambda_{k_i} (T-\tau)^\beta) q(\tau) d\tau = 0, \quad i \in \mathbb{N}. \tag{5}$$

Next we introduce the following restriction for q:

$$\exists\delta \in (0, T) \,:\, q(t) = 0,\; t \in (T - \delta, T).$$

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We use the asymptotics of Mittag-Leffler function

$$E_\beta(-z) = -\sum_{n=1}^{N} \frac{(-1)^n}{\Gamma(1-n\beta)z^n} + O\left(\frac{1}{z^{N+1}}\right) \quad \text{as} \quad z \to \infty, \quad N \in \mathbb{N}.$$

Then we obtain

$$0 = -\sum_{n=1}^{N} \frac{(-1)^n}{\lambda_{k_i}^n} \frac{1}{\Gamma(1-n\beta)} \int_0^{T-\delta} \frac{q(\tau)}{(T-\tau)^{n\beta}} d\tau + O\left(\frac{1}{\lambda_{k_i}^{N+1}}\right) \quad (7)$$

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Multiplying (7) by λ_{k_i} and passing to the limit $i \to \infty$ we obtain

$$\frac{1}{\Gamma(1-\beta)} \int_0^{T-\delta} \frac{q(\tau)}{(T-\tau)^{\beta}} d\tau = 0.$$

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Continuing this process we deduce the relations

$$\frac{1}{\Gamma(1-n\beta)} \int_0^{T-\delta} \frac{q(\tau)}{(T-\tau)^{n\beta}} d\tau = 0, \quad n \in \mathbb{N}. \quad (8)$$

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may not be useful.

Since the gamma function has poles at nonpositive integers, it holds

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Therefore, we consider separately two cases:

$$\beta\in(0,1)\setminus\mathbb{Q} \quad \text{and} \quad \beta\in(0,1)\cap\mathbb{Q}.$$

Firstly, let $\beta \in (0,1) \setminus \mathbb{Q}$. Then $\frac{1}{\Gamma(1-n\beta)} \neq 0$, $n \in \mathbb{N}$, hence

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The change of variable $s = \frac{1}{(T-\tau)^\beta}$ under the integral yields

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Therefore,

$$\int_{\frac{1}{T^\beta}}^{\frac{1}{\delta^\beta}} s^n v(s) ds = 0, \quad n \in \{0\} \cup \mathbb{N}, \quad v(s) = s^{-\frac{1}{\beta}} q(T - s^{-\frac{1}{\beta}}).$$

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$$\int_{\frac{1}{T^\beta}}^{\frac{1}{\delta^\beta}} \eta(s)v(s)ds = 0, \quad \text{for any } \eta \in L_2(\tfrac{1}{T^\beta}, \tfrac{1}{\delta^\beta}).$$

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So we pick up the basic relations with such n:

$$\frac{1}{\Gamma(1 - n' l_1 - \beta)} \int_0^{T-\delta} \frac{q(\tau)}{(T-\tau)^{n' l_1 + \beta}} d\tau = 0, \quad n' \in \mathbb{N}.$$

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The proof can be finished as in the previous case

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Assume that

$$\text{there exists a subsequence } k_j,\ j \in \mathbb{N} \text{ such that} \quad \varphi_{k_j} \neq 0,\ j \in \mathbb{N}, \quad \lim_{j\to\infty} \frac{f_{k_j}}{\varphi_{k_j}} = 0. \tag{9}$$

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Sufficient condition for (9):

If $\varphi \in L_2(\Omega) \setminus D((-L)^{\alpha})$ and $f \in D((-L)^{\alpha+\frac{m\gamma}{4}})$

for some $\alpha > 0$, $\gamma > 1$ then (9) holds.

Here $D((-L)^{\alpha}) = \{z \in L_2(\Omega)\ :\ \sum_{k=1}^{\infty} \mu_k^{2\alpha} |\langle z, v_k\rangle|^2 < \infty\}$.

Suppose that IP2 has a solution and consider the relations

$$\psi_{k_j} = \varphi_{k_j} E_\beta(-\lambda_{k_j} T^\beta) + f_{k_j} \int_0^T E_\beta(-\lambda_{k_j}(T-\tau)^\beta) q(\tau) d\tau, \quad j \in \mathbb{N}.$$

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Transform them to the form

$$\frac{\lambda_{k_j}\psi_{k_j}}{\varphi_{k_j}} = \lambda_{k_j} E_\beta(-\lambda_{k_j} T^\beta) + \frac{f_{k_j}}{\varphi_{k_j}} \int_0^T \lambda_{k_j} E_\beta(-\lambda_{k_j}(T-\tau)^\beta) q(\tau) d\tau, \, j \in \mathbb{N}$$

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The terms have the following behavior:

$$\underbrace{\frac{\lambda_{k_j}\psi_{k_j}}{\varphi_{k_j}}}_{\to M} = \underbrace{\lambda_{k_j} E_\beta(-\lambda_{k_j} T^\beta)}_{\to \frac{1}{\Gamma(1-\beta) T^\beta}} + \underbrace{\frac{f_{k_j}}{\varphi_{k_j}}}_{\to 0} \underbrace{\int_0^T \lambda_{k_j} E_\beta(-\lambda_{k_j}(T-\tau)^\beta) q(\tau) d\tau}_{\text{bounded}}.$$

Therefore, we obtain the equation

$$\frac{1}{\Gamma(1-\beta)T^{\beta}} = M \quad \text{where} \quad M = \lim_{j\to\infty} \frac{\lambda_{k_j}\psi_{k_j}}{\varphi_{k_j}}. \tag{10}$$

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In case $T \geq e^{-\gamma_*} \approx 0.561$, where $\gamma_* \approx 0.577$ is the Euler-Mascheroni constant,

the function $\frac{1}{\Gamma(1-\beta)T^{\beta}}$ is strictly monotonic for $\beta \in (0,1)$.

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Treatment of IP3 (there λ and q are unknown).

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$$\varphi_{k_j} \neq 0,\ j \in \mathbb{N}, \quad \lim_{j\to\infty} \frac{f_{k_j}}{\varphi_{k_j}} = 0.$$

Let us also recall that

$$\lambda_k = \lambda\mu_k,$$

where

μ_k is the eigenvalue of $-\Delta$ with Dirichlet boundary condition and
λ_k is the eigenvalue of $-\lambda\Delta$ with Dirichlet boundary condition.

Suppose that IP3 has a solution and consider the relations

$$\psi_{k_j} = \varphi_{k_j} E_\beta(-\lambda\mu_{k_j} T^\beta) + f_{k_j} \int_0^T E_\beta(-\lambda\mu_{k_j}(T-\tau)^\beta) q(\tau) d\tau, \quad j \in \mathbb{N}.$$

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Transform them to the form

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and take the limit $j \to \infty$.

Terms of this equation have the following behavior:

$$\underbrace{\frac{\mu_{k_j}\psi_{k_j}}{\varphi_{k_j}}}_{\to M_1} = \underbrace{\mu_{k_j} E_\beta(-\lambda\mu_{k_j} T^\beta)}_{\to \frac{1}{\lambda\Gamma(1-\beta)T^\beta}} + \underbrace{\frac{f_{k_j}}{\varphi_{k_j}}}_{\to 0} \underbrace{\int_0^T \mu_{k_j} E_\beta(-\lambda\mu_{k_j}(T-\tau)^\beta) q(\tau) d\tau}_{\text{bounded}}$$

Therefore, we obtain

$$\frac{1}{\lambda\Gamma(1-\beta)T^{\beta}} = M_1 \quad \text{where} \quad M_1 = \lim_{j\to\infty} \frac{\mu_{k_j}\psi_{k_j}}{\varphi_{k_j}}.$$

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This gives explicitly $\lambda = \frac{1}{M_1\Gamma(1-\beta)T^{\beta}}$.

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Treatment of IP4 (there β, λ and q are unknown).

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We obtain the equation

$$\frac{1}{\lambda\Gamma(1-\beta)T^\beta} = M_1 \quad \text{where} \quad M_1 = \lim_{j\to\infty} \frac{\mu_{k_j}\psi_{k_j}}{\varphi_{k_j}}.$$

Next we deduce the relation

$$\mu_{k_j}\left(\frac{\mu_{k_j}\psi_{k_j}}{\varphi_{k_j}} - M_1\right) = \mu_{k_j}\left(\mu_{k_j}E_\beta(-\lambda\mu_{k_j}T^\beta) - \frac{1}{\lambda\Gamma(1-\beta)T^\beta}\right)$$
$$+\frac{\mu_{k_j}f_{k_j}}{\varphi_{k_j}}\int_0^T \mu_{k_j}E_\beta(-\lambda\mu_{k_j}(T-\tau)^\beta)q(\tau)d\tau,\ j\in\mathbb{N}.$$

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Terms of this relation have the following behavior:

$$\underbrace{\mu_{k_j}\left(\frac{\mu_{k_j}\psi_{k_j}}{\varphi_{k_j}} - M_1\right)}_{\to M_2} = \underbrace{\mu_{k_j}\left(\mu_{k_j}E_\beta(-\lambda\mu_{k_j}T^\beta) - \frac{1}{\lambda\Gamma(1-\beta)T^\beta}\right)}_{\to -\frac{1}{\lambda^2\Gamma(1-2\beta)T^{2\beta}}}$$
$$+\underbrace{\frac{\mu_{k_j}f_{k_j}}{\varphi_{k_j}}}_{\to 0}\underbrace{\int_0^T \mu_{k_j}E_\beta(-\lambda\mu_{k_j}(T-\tau)^\beta)q(\tau)d\tau}_{\text{bounded}}$$

This implies

$$\frac{1}{\lambda^2\Gamma(1-2\beta)T^{2\beta}} = -M_2 \quad \text{where} \quad M_2 = \lim_{j\to\infty} \mu_{k_j}\left(\frac{\mu_{k_j}\psi_{k_j}}{\varphi_{k_j}} - M_1\right).$$

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We have the following system for β and λ:

$$\frac{1}{\lambda \Gamma(1-\beta) T^{\beta}} = M_1, \quad \frac{1}{\lambda^2 \Gamma(1-2\beta) T^{2\beta}} = -M_2.$$

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Eliminating λ we obtain

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The function $\frac{(\Gamma(1-\beta))^2}{\Gamma(1-2\beta)}$ is strictly monotonic for $\beta \in (0,1)$.

Hence β is uniquely determined.

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This implies that λ is also uniquely determined.

Finally, the uniqueness of q can be shown as in case IP1.

Let us formulate theorems. Define

$$\mathcal{Q} = \{q \,:\, q(t) = 0,\ t \in (T-\delta, T) \text{ for some } \delta \in (0,T),\ \ q \in L_2(0, T-\delta)\}$$

and

$$\mathcal{F} = \{f \in L_2(\Omega) \,:\, \text{there exists a subsequence } k_i \text{ of } \mathbb{N} \text{ such that } f_{k_i} \neq 0,\ i \in \mathbb{N}\}.$$

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Theorem 1

Let $f \in \mathcal{F}$ and $\varphi = \psi = 0$. If $\mathrm{IP1}$ has a solution $q \in \mathcal{Q}$ then $q = 0$.

Theorem 2

Let $f \in \mathcal{F}$ and $T \geq e^{-\gamma_*}$, where γ_* is the Euler-Mascheroni constant. Moreover, assume that

$$\varphi \in L_2(\Omega), \quad \text{there exists a subsequence } k_j,\ j \in \mathbb{N} \text{ such that}$$

$$\varphi_{k_j} \neq 0,\ j \in \mathbb{N}, \quad \lim_{j\to\infty} \frac{f_{k_j}}{\varphi_{k_j}} = 0.$$

If IP2 *has solutions* (β, q), $(\tilde{\beta}, \tilde{q}) \in (0,1) \times \mathcal{Q}$ *then* $\beta = \tilde{\beta}$, $q = \tilde{q}$.

Theorem 3

Let $f \in \mathcal{F}$. *Moreover, assume that*

$$\varphi \in L_2(\Omega), \quad \text{there exists a subsequence } k_j,\ j \in \mathbb{N} \text{ such that}$$

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If IP3 *has solutions* (λ, q), $(\tilde{\lambda}, \tilde{q}) \in (0, \infty) \times \mathcal{Q}$ *then* $\lambda = \tilde{\lambda}$, $q = \tilde{q}$.

Theorem 4

Let $f \in \mathcal{F}$. *Moreover, assume that*

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If IP4 *has solutions* (β, λ, q), $(\tilde{\beta}, \tilde{\lambda}, \tilde{q}) \in (0,1) \times (0,\infty) \times \mathcal{Q}$ *then* $\beta = \tilde{\beta}$, $\lambda = \tilde{\lambda}$, $q = \tilde{q}$.

Thank you for the attention!