Analysis on radial complex-valued functions of p-adic arguments

Anatoly N. Kochubei
Institute of Mathematics,
National Academy of Sciences of Ukraine,
Tereshchenkivska 3, Kiev, 01024 Ukraine,
E-mail: kochubei@imath.kiev.ua

Preliminaries

1. p-Adic numbers

Let p be a prime number. The field of p-adic numbers is the completion $\mathbb{Q}_p$ of the field $\mathbb{Q}$ of rational numbers, with respect to the absolute value $|x|_p$ defined by setting $|0|_p = 0$,

$$|x|_p = p^{-\nu} \text{ if } x = p^{\nu}\frac{m}{n},$$

where $\nu, m, n \in \mathbb{Z}$, and m, n are prime to p. Example: $|p|_p = p^{-1}$.

$\mathbb{Q}_p$ is a locally compact topological field.

Note that by Ostrowski's theorem there are no absolute values on $\mathbb{Q}$, which are not equivalent to the "Euclidean" one, or one of $|\cdot|_p$. We denote $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : \ |x|_p \leq 1\}$. $\mathbb{Z}_p$, as well as all balls in $\mathbb{Q}_p$, is simultaneously open and closed.

The absolute value $|x|_p$, $x \in \mathbb{Q}_p$, has the following properties:

$$|x|_p = 0 \text{ if and only if } x = 0;$$
$$|xy|_p = |x|_p \cdot |y|_p;$$
$$|x + y|_p \leq \max(|x|_p, |y|_p).$$

The latter property called the ultra-metric inequality (or the non-Archimedean property) implies the total disconnectedness of $\mathbb{Q}_p$ in the topology determined by the metric $|x - y|_p$, as well as many unusual geometric properties (Example: *two balls either do not intersect, or one of them is contained in another*). Note also the following consequence of the ultra-metric inequality:

$$|x + y|_p = \max(|x|_p, |y|_p) \quad \text{if } |x|_p \neq |y|_p.$$

The absolute value $|x|_p$ takes the discrete set of non-zero values p^N, $N \in \mathbb{Z}$. If $|x|_p = p^N$, then x admits a (unique) canonical representation

$$x = p^{-N}\left(x_0 + x_1 p + x_2 p^2 + \cdots\right),$$

where $x_0, x_1, x_2, \ldots \in \{0, 1, \ldots, p-1\}$, $x_0 \neq 0$. The series converges in the topology of $\mathbb{Q}_p$. For example,

$$-1 = (p-1) + (p-1)p + (p-1)p^2 + \cdots, \quad |-1|_p = 1.$$

The canonical representation shows the hierarchical structure of $\mathbb{Q}_p$.

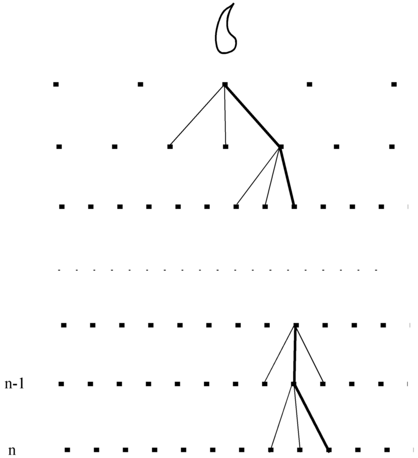

Figure: Structure of the p-adic tree.

Khrennikov et al (2016) - p-adic model of a porous medium.

Applications:

- leakage of a liquid through a porous medium;
- physics of hierarchical structures (for example, dynamics of protein molecules).

Mathematical developments:

Since rational numbers are used in almost all branches of mathematics, Ostrowski's theorem (1916) has divided mathematics into two parts, the classical one and the non-Archimedean one, based on p-adic numbers.
By this time, there exist non-Archimedean counterparts of almost all branches of mathematics.

2. Local fields. Let K be a non-Archimedean local field, that is a non-discrete totally disconnected locally compact topological field. It is well known that K is isomorphic either to a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers (if K has characteristic 0), or to the field of formal Laurent series with coefficients from a finite field, if K has a positive characteristic.

Any local field K is endowed with an absolute value $|\cdot|_K$, such that $|x|_K = 0$ if and only if $x = 0$, $|xy|_K = |x|_K \cdot |y|_K$, $|x+y|_K \le \max(|x|_K, |y|_K)$. Denote $O = \{x \in K: \ |x|_K \le 1\}$, $P = \{x \in K: \ |x|_K < 1\}$. O is a subring of K, and P is an ideal in O containing such an element β that $P = \beta O$. The quotient ring O/P is actually a finite field; denote by q its cardinality. We will always assume that the absolute value is normalized, that is $|\beta|_K = q^{-1}$. The normalized absolute value takes the values q^N, $N \in \mathbb{Z}$. Note that for $K = \mathbb{Q}_p$ we have $\beta = p$ and $q = p$; the p-adic absolute value is normalized.

Denote by $S \subset O$ a complete system of representatives of the residue classes from O/P. Any nonzero element $x \in K$ admits the canonical representation in the form of the convergent series

$$x = \beta^{-n} \left(x_0 + x_1 \beta + x_2 \beta^2 + \cdots \right)$$

where $n \in \mathbb{Z}$, $|x|_K = q^n$, $x_j \in S$, $x_0 \notin P$. For $K = \mathbb{Q}_p$, one may take $S = \{0, 1, \ldots, p-1\}$.

The additive group of any local field is self-dual, that is if χ is a fixed non-constant complex-valued additive character of K, then any other additive character can be written as $\chi_a(x) = \chi(ax)$, $x \in K$, for some $a \in K$. Below we assume that χ is a rank zero character, that is $\chi(x) \equiv 1$ for $x \in O$, while there exists such an element $x_0 \in K$ that $|x_0|_K = q$ and $\chi(x_0) \neq 1$.

Fourier transform:

$$\widetilde{f}(\xi) = \int\limits_K \chi(x\xi) f(x)\, dx, \quad \xi \in K,$$

Inverse transform:

$$f(x) = \int\limits_K \chi(-x\xi) \widetilde{f}(\xi)\, d\xi.$$

Test functions from $\mathcal{D}(K)$: locally constant functions with compact supports.
$\mathcal{D}'(K)$ - Bruhat-Schwartz distributions.

Vladimirov operator:

$$\left(D^{\alpha}\varphi\right)(x)=\mathcal{F}^{-1}\left[|\xi|_K^{\alpha}(\mathcal{F}(\varphi))(\xi)\right](x),\alpha>0,$$

$\varphi\in\mathcal{D}(K)$.
The operator D^{α} can also be represented as a hypersingular integral operator:

$$\left(D^{\alpha}\varphi\right)(x)=\frac{1-q^{\alpha}}{1-q^{-\alpha-1}}\int\limits_K |y|_K^{-\alpha-1}[\varphi(x-y)-\varphi(x)]\,dy.$$

This expression makes sense for wider classes of functions.
As an operator in L^2, D^{α} is selfadjoint, nonnegative with pure point spectrum $\{q^{\alpha N},\ N\in\mathbb{Z}\}$ of *infinite* multiplicity.

Right inverse ($\alpha > 0$):

$$\left(D^{-\alpha}\varphi\right)(x) = (f_\alpha * \varphi)(x) = \frac{1-q^{-\alpha}}{1-q^{\alpha-1}} \int\limits_K |x-y|_K^{\alpha-1}\varphi(y)\,dy, \quad \varphi \in \mathcal{D}(K),$$

and

$$\left(D^{-1}\varphi\right)(x) = \frac{1-q}{q\log q} \int\limits_K \log|x-y|_K \varphi(y)\,dy.$$

Then $D^\alpha D^{-\alpha} = I$ on $\mathcal{D}(K)$, if $\alpha \neq 1$. This property remains valid on $\Phi(K)$ also for $\alpha = 1$. Here

$$\Phi(K) = \left\{ \varphi \in \mathcal{D}(K) : \int\limits_K \varphi(x)\,dx = 0 \right\}.$$

is the so-called Lizorkin space.

Vladimirov Operator on Radial Functions

a) *Radial eigenfunctions*
Let $u(x) = \psi(|x|) \in L_2(K)$,

$$D^\alpha u = \lambda u, \quad \lambda = q^{\alpha N},\ N \in \mathbb{Z},$$

and u is not identically zero. Then

$$u(x) = \begin{cases} cq^N(1 - q^{-1}), & \text{if } |x| \le q^{-N}; \\ -cq^{N-1}, & \text{if } |x| = q^{-N+1}; \\ 0, & \text{if } |x| > q^{-N+1}. \end{cases}$$

It is shown that $u \in \Phi(K)$.
The only radial eigenfunction u with $u(0) = 1$ (an analog of the function $t \to e^{-\lambda t}$, $t \in \mathbb{R}$) corresponds to $c = q^{-N}(1 - q^{-1})^{-1}$.
On the other hand, if $u(0) = 0$, then $c = 0$.

This is the cornerstone of the theory of non-Archimedean wave equation.

b) *Explicit formula*

Lemma

If a function $u = u(|x|_K)$ is such that

$$\sum_{k=-\infty}^{m} q^k \left|u(q^k)\right| < \infty, \quad \sum_{l=m}^{\infty} q^{-\alpha l} \left|u(q^l)\right| < \infty,$$

for some $m \in \mathbb{Z}$, then for each $n \in \mathbb{Z}$ the hypersingular integral for $D^\alpha \varphi$ with $\varphi(x) = u(|x|_K)$ exists for $|x|_K = q^n$, depends only on $|x|_K$, and

$$(D^\alpha u)(q^n) = d_\alpha \left(1 - \frac{1}{q}\right) q^{-(\alpha+1)n} \sum_{k=-\infty}^{n-1} q^k u(q^k)$$
$$+ q^{-\alpha n - 1} \frac{q^\alpha + q - 2}{1 - q^{-\alpha-1}} u(q^n) + d_\alpha \left(1 - \frac{1}{q}\right) \sum_{l=n+1}^{\infty} q^{-\alpha l} u(q^l)$$

where $d_\alpha = \dfrac{1 - q^\alpha}{1 - q^{-\alpha-1}}$.

The regularized integral

$$\boxed{(I^{\alpha}\varphi)(x) = (D^{-\alpha}\varphi)(x) - (D^{-\alpha}\varphi)(0).}$$

This is defined initially for $\varphi \in \mathcal{D}(K)$. In fact,

$$(I^{\alpha}\varphi)(x) = \frac{1-q^{-\alpha}}{1-q^{\alpha-1}} \int\limits_{|y|_K \le |x|_K} \left(|x-y|_K^{\alpha-1} - |y|_K^{\alpha-1}\right)\varphi(y)\,dy, \quad \alpha \neq 1,$$

and

$$(I^{1}\varphi)(x) = \frac{1-q}{q\log q} \int\limits_{|y|_K \le |x|_K} \left(\log|x-y|_K - \log|y|_K\right)\varphi(y)\,dy.$$

In contrast to $D^{-\alpha}$, the integrals are taken, for each fixed $x \in K$, over bounded sets.

MAIN LEMMA

Suppose that

$$\sum_{k=-\infty}^{m} \max\left(q^k, q^{\alpha k}\right) \left|u(q^k)\right| < \infty, \quad \text{if } \alpha \neq 1,$$

$$\sum_{k=-\infty}^{m} |k| q^k \left|u(q^k)\right| < \infty, \quad \text{if } \alpha = 1,$$

for some $m \in \mathbb{Z}$. Then $I^\alpha u$ exists, it is a radial function, and for any $x \neq 0$,

$$(I^\alpha u)(|x|_K) = q^{-\alpha}|x|_K^\alpha u(|x|_K) + \frac{1-q^{-\alpha}}{1-q^{\alpha-1}} \int\limits_{|y|_K<|x|_K} \left(|x|_K^{\alpha-1} - |y|_K^{\alpha-1}\right) u(|y|_K)\, dy, \quad \alpha \neq 1,$$

and

$$(I^1 u)(|x|_K) = q^{-1}|x|_K u(|x|_K) + \frac{1-q}{q\log q} \int\limits_{|y|_K<|x|_K} (\log|x|_K - \log|y|_K)\, u(|y|_K)\, dy.$$

Proposition ("right inverse")

Suppose that for some $m \in \mathbb{Z}$,

$$\sum_{k=-\infty}^{m} \max\left(q^k, q^{\alpha k}\right) \left|v(q^k)\right| < \infty, \quad \sum_{l=m}^{\infty} \left|v(q^l)\right| < \infty,$$

if $\alpha \neq 1$, *and*

$$\sum_{k=-\infty}^{m} |k| q^k \left|v(q^k)\right| < \infty, \quad \sum_{l=m}^{\infty} l \left|v(q^l)\right| < \infty,$$

if $\alpha = 1$. *Then there exists* $(D^\alpha I^\alpha v)(|x|_K) = v(|x|_K)$ *for any* $x \neq 0$.

Proposition ("left inverse")

Suppose that $u(0) = 0$,

$$|u(q^n)| \leq Cq^{dn}, \quad n \leq 0;$$

$$|u(q^n)| \leq Cq^{hn}, \quad n \geq 0,$$

where $d > \max(0, \alpha - 1)$, $0 \leq h < \alpha$, *and* $h < \alpha - 1$, *if* $\alpha > 1$. *Then the function* $w = D^\alpha u$ *satisfies, for any* $m \in \mathbb{Z}$, *the inequalities*

$$\sum_{k=-\infty}^{m} \max(q^k, q^{\alpha k}) \left| w(q^k) \right| < \infty, \quad \sum_{l=m}^{\infty} \left| w(q^l) \right| < \infty, \quad \alpha \neq 1;$$

$$\sum_{k=-\infty}^{m} |k| \cdot q^k \left| w(q^k) \right| < \infty, \quad \sum_{l=m}^{\infty} |l| \cdot \left| w(q^l) \right| < \infty, \quad \alpha = 1.$$

Moreover, $I^\alpha D^\alpha u = u$.

Corollary

Let $v = v_0 + u$ where v_0 is a constant, u satisfies the conditions of the above Proposition. Then $I^\alpha D^\alpha v = v - v_0$.

EXAMPLE: the simplest Cauchy problem

$$D^\alpha u(|x|_K) = f(|x|_K), \quad u(0) = 0,$$

where f is a continuous function, such that

$$\sum_{l=m}^{\infty} \left| f(q^l) \right| < \infty, \text{ if } \alpha \neq 1, \text{ or } \sum_{l=m}^{\infty} l \left| f(q^l) \right| < \infty, \text{ if } \alpha = 1.$$

The unique strong solution is $u = I^\alpha f$. Therefore on radial functions, the operators D^α and I^α behave like the Caputo-Dzhrbashyan fractional derivative and the Riemann-Liouville fractional integral of real analysis.

(COUNTER)-EXAMPLE: Let $f(|x|_K) \equiv 1$, $x \in K$. Then $(I^\alpha f)(|x|_K) \equiv 0$.

THE CAUCHY PROBLEM

a) Local solvability. Consider the equation

$$(D^{\alpha}u)(|t|_K) = f(|t|_K, u(|t|_K)), \quad 0 \neq t \in K, \tag{1}$$

with the initial condition

$$u(0) = u_0 \tag{2}$$

where the function $f: \ q^{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ satisfies the conditions

$$|f(|t|_K, x)| \leq M; \tag{3}$$

$$|f(|t|_K, x) - f(|t|_K, y)| \leq F|x - y|, \tag{4}$$

for all $t \in K, x, y \in \mathbb{R}$.

With the problem (1)-(2) we associate the integral equation

$$u(|t|_K) = u_0 + I^\alpha f(|\cdot|_K, u(|\cdot|_K))(|t|_K). \tag{5}$$

Note that, by the definition of I^α, in order to compute $(I^\alpha \varphi)(|t|_K)$ for $|t|_K \leq q^m$ ($m \in \mathbb{Z}$), one needs to know the function φ in the same ball $|t|_K \leq q^m$. Therefore local solutions of the equation (5) make sense, in contrast to solutions of (1).
We call a solution u of (5), if it exists, *a mild solution* of the Cauchy problem (1)-(2). By the above Corollary, a solution u of (1)-(2), such that $u - u_0$ satisfies the conditions of Proposition, is a mild solution.

Theorem

Under the assumptions (3),(4), the problem (1)-(2) has a unique local mild solution, that is the integral equation (5) has a solution $u(|t|_K)$ defined for $|t|_K \leq q^N$ where $N \in \mathbb{Z}$ is sufficiently small, and another solution $\overline{u}(|t|_K)$, if it exists, coincides with u for $|t|_K \leq q^{\overline{N}}$ where $\overline{N} \leq N$.

b) Extension of solutions. Let us study the possibility to continue the local solution constructed in Theorem 1 to a solution of the integral equation (5) defined for all $t \in K$.
Suppose that the conditions of Theorem 1 are satisfied, and we obtained a local solution $u(|t|_K)$, $|t|_K \leq q^N$, $N \in \mathbb{Z}$. Let $\alpha \neq 1$. In order to find a solution for $|t|_K = q^{N+1}$, we have to solve the equation

$$u(q^{N+1}) = u_0 + v_0^{(N)} + q^{\alpha N} f(q^{N+1}, u(q^{N+1})) \tag{6}$$

where

$$v_0^{(N)} = \frac{1 - q^{-\alpha}}{1 - q^{\alpha - 1}} \int\limits_{|y|_K \leq q^N} \left(q^{(N+1)(\alpha-1)} - |y|_K^{\alpha-1} \right) f(|y|_K, u(|y|_K))\, dy. \tag{7}$$

is a known constant. A similar equation can be written for $\alpha = 1$. Then the above procedure, if it is successful, is repeated for all $l > N$.

Theorem

Suppose that the conditions of local existence theorem are satisfied, as well as the following Lipschitz condition:

$$|f(q^l, x) - f(q^l, y)| \leq F_l |x - y|, \quad x, y \in \mathbb{R}, \ l \in \mathbb{Z}, \tag{8}$$

where $0 < F_l < q^{-\alpha l}$ *for each* $l \in \mathbb{Z}$*. Then a local solution of the equation (5) admits a continuation to a global solution defined for all* $t \in K$*.*

c) From an integral equation to a differential one. Let us study conditions, under which the above continuation procedure leads to a solution of the problem (1)-(2). As before, we assume the conditions (3),(4) and (8). In addition, we will assume that

$$|f(q^l, x)| \leq Cq^{-\beta l}, \quad l \geq 1, \quad \text{for all } x \in \mathbb{R}, \tag{9}$$

where $\beta > \alpha$.

Theorem

Under the assumptions (3),(4), (8) and (9), the mild solution obtained by the iteration process with subsequent continuation, satisfies the equation (1).

Some Publications

1. A. N. Kochubei, A non-Archimedean wave equation, *Pacif. J. Math.* **235** (2008), 245–261.
2. A. N. Kochubei, Radial solutions of non-Archimedean pseudodifferential equations, *Pacif. J. Math.* **269** (2014), 355–369.
3. A. N. Kochubei, Nonlinear pseudo-differential equations for radial real functions on a non-Archimedean field. *J. Math. Anal. Appl.* **483** (2020), no. 1, Article 123609.
4. A. N. Kochubei, Non-Archimedean Radial Calculus: Volterra Operator and Laplace Transform, arXiv 2005.11166.