Noncommutative Fourier Series on $\Gamma\backslash SE(2)$

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Pseudo-differential conference
BE HARMONIC WITH ANALYSIS
$\int \mathrm{Tr}(\pi(x)\sigma(x,\pi)\hat{f}(\pi))d\mu(\pi)$
Make a difference: pseudo-differentiate
integrate with mind but pseudo-differentiate with heart

Abstract

This work begins with a systematic study of abstract noncommutative Fourier series on $\Gamma\backslash SE(2)$, where Γ is a discrete and co-compact subgroup of $SE(2)$, the group of handedness-preserving Euclidean isometries of the Euclidean plane.

Introduction

The 2D special Euclidean group, usually denoted as $SE(2)$, is one of the simplest examples of a noncommutative, noncompact real finite dimensional Lie group. The right coset space of discrete and co-compact subgroups in $SE(2)$, such as $\mathbb{Z}^2\backslash SE(2)$, appears as the configuration space in many recent applications in computational science and engineering including computer vision, robotics, mathematical crystallography, computational biology, and material science [2, 3, 4].

Preliminaries and Notation

The 2D special Euclidean group, $SE(2)$, is the semidirect product of $\mathbb{R}^2$ with the 2D special orthogonal group $SO(2)$. We denote elements $g \in SE(2)$ as $g = (\mathbf{x}, \mathbf{R})$ where $\mathbf{x} \in \mathbb{R}^2$ and $\mathbf{R} \in SO(2)$.
The classical Fourier Plancherel formula on the group $SE(2)$ is

$$\int_{SE(2)} |f(g)|^2 dg = \int_0^\infty \|\widehat{f}(p)\|_{\mathrm{HS}}^2 p dp,$$

and also the noncommutative Fourier reconstruction formula is

$$f(g) = \int_0^\infty \mathrm{tr}\left[\widehat{f}(p)U_p(g)\right] p dp, \quad (1)$$

for $f \in L^1 \cap L^2(SE(2))$ and $g \in SE(2)$, where for each $p > 0$, the irreducible representation $U_p : SE(2) \to \mathcal{U}(L^2(\mathbb{S}^1))$ is

$$[U_p(g)\varphi](\mathbf{u}) := e^{-ip\langle \mathbf{u},\mathbf{t}\rangle}\varphi(\mathbf{R}^T\mathbf{u}), \quad (2)$$

for all $g = (\mathbf{t}, \mathbf{R}) \in SE(2)$, $\varphi \in L^2(\mathbb{S}^1)$, $\mathbf{u} \in \mathbb{S}^1$, and the Fourier transform of each $f \in L^1(SE(2))$ at $p > 0$ is given by

$$\widehat{f}(p) := \int_{SE(2)} f(g)U_p(g^{-1})dg. \quad (3)$$

The standard orthonormal basis for the Hilbert function space $L^2(\mathbb{S}^1)$ is $\mathcal{B} := \{\mathbf{e}_k : k \in \mathbb{Z}\}$, where for each $k \in \mathbb{Z}$, $\mathbf{e}_k : \mathbb{S}^1 \to \mathbb{C}$ is given by

$$\mathbf{e}_k(\mathbf{u}_\alpha) = \mathbf{e}_k(\cos\alpha, \sin\alpha) := e^{ik\alpha}.$$

For each $g \in SE(2)$ and $p > 0$, the matrix elements of the operator $U_p(g)$ with respect to the basis $\mathcal{B}$, are expressed as

$$\mathrm{u}_{mn}(g, p) = \langle U_p(g)\mathbf{e}_m, \mathbf{e}_n\rangle_{L^2(\mathbb{S}^1)}, \quad (4)$$

for each $m, n \in \mathbb{Z}$, see [1].

State of the problem

- Let Γ be a discrete co-compact subgroup of $SE(2)$.
- Suppose μ is the finite $SE(2)$-invariant measure on the right coset space $\Gamma\backslash SE(2)$ which is normalized with respect to Weil's formula given by

$$\int_{\Gamma\backslash SE(2)} \tilde{f}(\Gamma g)d\mu(\Gamma g) = \int_{SE(2)} f(g)dg, \quad (5)$$

for $f \in L^1(SE(2))$, where

$$\tilde{f}(\Gamma g) := \sum_{\gamma\in\Gamma} f(\gamma \circ g), \quad (6)$$

for $g \in SE(2)$.
- Let $\varphi = \tilde{f} \in L^1(\Gamma\backslash SE(2), \mu)$ with $f \in L^1(SE(2))$ be given.
- How to expand φ using coefficients of Fourier matrix elements of f on $SE(2)$ given by (3)?

Theorem (GF-Chirikjian)

Let Γ be a discrete co-compact subgroup of $SE(2)$ and $\mathcal{E}(\Gamma) := \{\psi_\ell : \Gamma\backslash SE(2) \to \mathbb{C} \mid \ell \in \mathbb{I}\}$ be a (discrete) orthonormal basis for the Hilbert function space $L^2(\Gamma\backslash SE(2), \mu)$. Let $f \in L^1 \cap L^2(SE(2))$ such that $\tilde{f} \in L^2(\Gamma\backslash SE(2), \mu)$. We then have

$$\langle \tilde{f}, \psi_\ell\rangle = \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \int_0^\infty \widehat{f}(p)_{nm} Q^\ell(p)_{mn} p dp, \quad (7)$$

where

$$Q^\ell(p)_{mn} := \int_{SE(2)} \mathrm{u}_{mn}(g, p)\overline{\psi_\ell(\Gamma g)}dg,$$

and

$$\widehat{f}(p)_{nm} = \langle \widehat{f}(p)\mathbf{e}_n, \mathbf{e}_m\rangle,$$

for $p > 0$, $\ell \in \mathbb{I}$, and $m, n \in \mathbb{Z}$.

Proposition (GF-Chirikjian)

Let Γ be a discrete co-compact subgroup of $SE(2)$ and $\mathcal{E}(\Gamma) := \{\psi_\ell : \Gamma\backslash SE(2) \to \mathbb{C} \mid \ell \in \mathbb{I}\}$ be a (discrete) orthonormal basis for the Hilbert function space $L^2(\Gamma\backslash SE(2), \mu)$. Let $f \in L^1(SE(2))$ such that $\widetilde{|f|} \in L^2(\Gamma\backslash SE(2), \mu)$. We then have

$$\langle \tilde{f}, \psi_\ell\rangle = \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \int_0^\infty \widehat{f}(p)_{nm} Q^\ell(p)_{mn} p dp, \quad (8)$$

Let $f \in L^1(SE(2))$ and $\psi \in L^1(\Gamma\backslash SE(2), \mu)$. We then define the convolution of f with ψ as the function $\psi \oslash f : \Gamma\backslash SE(2) \to \mathbb{C}$ via

$$(\psi \oslash f)(\Gamma g) := \int_{SE(2)} \psi(\Gamma h)f(h^{-1} \circ g)dh, \quad (9)$$

for $g \in SE(2)$.

Theorem (GF-Chirikjian)

Let Γ be a discrete co-compact subgroup of $SE(2)$ and $\mathcal{E}(\Gamma) := \{\psi_\ell : \Gamma\backslash SE(2) \to \mathbb{C} \mid \ell \in \mathbb{I}\}$ be a (discrete) orthonormal basis for the Hilbert function space $L^2(\Gamma\backslash SE(2), \mu)$. Let $f_k \in L^1(SE(2))$ with $k \in \{1, 2\}$ such that $\widetilde{|f_1|} \in L^2(\Gamma\backslash SE(2), \mu)$. We then have

$$\langle \widetilde{f_1} \oslash f_2, \psi_\ell\rangle = \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \int_0^\infty \widehat{f_1}(p)_{nk}\widehat{f_2}(p)_{km} Q^\ell(p)_{mn} p dp. \quad (10)$$

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