L^p-L^q boundedness of pseudo-differential operators on smooth manifolds and its applications to nonlinear equations

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Abstract

We present the boundedness of global pseudo-differential operators on smooth manifolds obtained in [3]. By using the notion of global symbol [6, 7] we extend a classical condition of Hörmander type to guarantee the L^p-L^q-boundedness of global operators. First we investigate L^p-boundedness of pseudo-multipliers in view of the Hörmander-Mihlin condition. Later, we concentrate to settle L^p-L^q boundedness of the Fourier multipliers and pseudo-differential operators for the range $1 < p \leq 2 \leq q < \infty$. Finally, we present applications of our boundedness theorems to the well-posedness properties of different types of nonlinear partial differential equations.

Introduction

- The boundedness results for pseudo-differential operators in their own right are interesting and important but also these serve as crucial tools to tackle several important problems of mathematics, in particular, of nonlinear PDEs. We refer [8, 7] and references therein for available extensive literature on these topics.
- Our main purpose is to extend seminal and classical result of L. Hörmander on L^p-L^q boundedness for pseudo-differential operators [5] to manifolds using the nonharmonic Fourier analysis and global quantization developed the last two authors [6] (see also [4]). Recently, L^p-L^q boundedness results for Fourier multipliers have been established in [1, 2] in the context of unimodular locally compact groups.
- To prove L^p-L^q boundedness of global operators we establish and apply the Paley-inequality and Hausdorff-Young-Paley inequality in the setting of nonharmonic analysis.

Fourier analysis associated to a model operator L on $\overline{M}$

- Let L be a pseudo-differential operator (need not be elliptic or self-adjoint) of order m on the interior M of a smooth manifold with boundary $\overline{M}$ in the sense of Hörmander.
- Assume that some boundary conditions (BC) are fixed and lead to a discrete spectrum with a family of eigenfunctions yielding a Riesz basis in $L^2(M)$.
- The discrete spectrum is $\{\lambda_\xi \in \mathbb{C} : \xi \in \mathcal{I}\}$ of L with corresponding eigenfunctions in $L^2(M)$ denoted by u_ξ which satisfy the boundary conditions (BC).
- The conjugate spectral problem is
$$L^* v_\xi = \overline{\lambda_\xi} v_\xi \text{ in } M, \text{ for all } \xi \in \mathcal{I},$$
which we equip with the conjugate boundary conditions (BC)*. We further assume that the functions u_ξ, v_ξ are normalised and the systems $\{u_\xi\}_{\xi\in\mathcal{I}}$ and $\{v_\xi\}_{\xi\in\mathcal{I}}$ are bi-orthogonal.
- The space $C_L^\infty(\overline{M}) := \cap_{k=1}^\infty \mathrm{Dom}(L^k)$, where $\mathrm{Dom}(L^k) := \{f \in L^2(M) \mid L^j f \in \mathrm{Dom}(L),\ j = 0, 1, \cdots, k-1\}$, so that the boundary condition (BC) are satisfied by the operators L^j.
- The L-Fourier transform of $f \in C_L^\infty(\overline{M})$ is defined by
$$(\mathcal{F}_L f)(\xi) := \widehat{f}(\xi) := \int_{\overline{M}} f(x)\overline{v_\xi(x)}\, dx.$$
- We refer to [6, 4] for more details.

L-pseudo-differential operators on manifolds with boundary

- An L-pseudo-differential operator is a continuous linear operator $A : C_L^\infty(\overline{M}) \to C_L^\infty(\overline{M})$, defined by
$$Af(x) = \sum_{\xi\in\mathcal{I}} u_\xi(x) m(x,\xi)\widehat{f}(\xi),\ f \in \mathrm{Dom}(A).$$
In this case, the function $m : \overline{M} \times \mathcal{I} \to \mathbb{C}$, is called the L-symbol associated with A.
- Denote by L° the densely defined operator given by $L^\circ u_\xi = \overline{\lambda_\xi} u_\xi, \quad \xi \in \mathcal{I}$.
- An L-pseudo-differential operator A is called L-pseudo-multiplier if there exists a continuous function $\tau_m : \overline{M} \times \mathbb{R} \to \mathbb{C}$, such that for every $\xi \in \mathcal{I}$, and $x \in \overline{M}$, we have $m(x,\xi) = \tau_m(x, |\lambda_\xi|)$. So, A is given by
$$Af(x) \equiv \tau_m(x, \sqrt{L^\circ L})f(x) := \sum_{\xi\in\mathcal{I}} u_\xi(x)\tau_m(x, |\lambda_\xi|)\widehat{f}(\xi).$$

Assumptions for H-M theorem

- There exist $-\infty < \gamma_p^{(1)}, \gamma_p^{(2)} < \infty$, satisfying
$$\|u_\xi\|_{L^p(M)} \lesssim |\lambda_\xi|^{\gamma_p^{(1)}}, \quad \|v_\xi\|_{L^{p'}(M)} \lesssim |\lambda_\xi|^{\gamma_p^{(2)}},$$
for $1 \leq p \leq \infty$.
- The operator $\sqrt{L^\circ L}$ satisfies the Weyl-eigenvalue counting formula
$$N(\lambda) := \sum_{\xi\in\mathcal{I}:|\lambda_\xi|\leq\lambda} = O(\lambda^Q),\ \lambda \to \infty,$$
where $Q > 0$. If $Q' > Q$, then $N(\lambda) = O(\lambda^{Q'})$, $\lambda \to \infty$, so that we assume that Q is the smallest real number with this property.

L^p-boundedness of L-pseudo-multipliers operators on $\overline{M}$

Theorem (Hörmander-Mihlin (H-M) theorem for pseudo-multipliers)

Let $\overline{M}$ be a smooth manifold with boundary and let $A : C_L^\infty(\overline{M}) \to C_L^\infty(\overline{M})$ be an L-pseudo-multiplier. Let us assume that τ_m satisfies the following Hörmander condition,
$$\|\tau_m\|_{l.u.\mathcal{H}^s} = \sup_{r>0, x\in\overline{M}} r^{(s-\frac{Q_m}{2})}\|\langle\cdot\rangle^s \mathscr{F}[\tau_m(x,\cdot)\psi(r^{-1}\cdot)]\|_{L^2(\mathbb{R})} < \infty,$$
for $s > \max\{1/2, \gamma_p + Q + (Q_m/2)\}$. Then $A \equiv T_m : L^p(\overline{M}) \to L^p(\overline{M})$ extends to a bounded linear operator for all $1 < p < \infty$.

L^p-L^q-boundedness of L-Fourier multipliers operators on $\overline{M}$

Theorem (Hörmander theorem for L-Fourier multipliers)

Let $1 < p \leq 2 \leq q < \infty$ and assume that
$$\sup_{\xi\in\mathcal{I}} \left(\frac{\|v_\xi\|_{L^\infty(\overline{M})}}{\|u_\xi\|_{L^\infty(\overline{M})}}\right) < \infty \quad \text{and} \quad \sup_{\xi\in\mathcal{I}} \left(\frac{\|u_\xi\|_{L^\infty(\overline{M})}}{\|v_\xi\|_{L^\infty(\overline{M})}}\right) < \infty.$$
Suppose that $A : C_L^\infty(\overline{M}) \to C_L^\infty(\overline{M})$ is an L-Fourier multiplier with L-symbol $\sigma_{A,L}$ on $\overline{M}$, that is, A satisfies
$$\mathcal{F}_L(Af)(\xi) = \sigma_{A,L}(\xi)\mathcal{F}_L f(\xi), \quad \text{for all } \xi \in \mathcal{I},$$
where $\sigma_{A,L} : \mathcal{I} \to \mathbb{C}$ is a function. Then we have
$$\|A\|_{\mathscr{B}(L^p(\overline{M}), L^q(\overline{M}))} \lesssim \sup_{s>0} s \left(\sum_{\substack{\xi\in\mathcal{I} \\ |\sigma_{A,L}(\xi)|\geq s}} \max\{\|u_\xi\|^2_{L^\infty(\overline{M})}, \|v_\xi\|^2_{L^\infty(\overline{M})}\} \right)^{\frac{1}{p}-\frac{1}{q}}$$

L^p-L^q-boundedness of L-pseudo-differential operators on $\overline{M}$

Theorem (Hörmander theorem for L-pseudo-differential operators)

Let $1 < p \leq 2 \leq q < \infty$ and assume that
$$\sup_{\xi\in\mathcal{I}} \left(\frac{\|v_\xi\|_{L^\infty(\overline{M})}}{\|u_\xi\|_{L^\infty(\overline{M})}}\right) < \infty \quad \text{and} \quad \sup_{\xi\in\mathcal{I}} \left(\frac{\|u_\xi\|_{L^\infty(\overline{M})}}{\|v_\xi\|_{L^\infty(\overline{M})}}\right) < \infty.$$
Suppose that $A : C_L^\infty(\overline{M}) \to C_L^\infty(\overline{M})$ is a continuous linear operators with L-symbol $\sigma_{A,L} : \overline{M} \times \mathcal{I} \to \mathbb{C}$, where $\overline{M}$ is a compact manifold (with or without boundary), satisfying
$$\|\sigma_{A,L}\|_{(\beta)} := \sup_{s>0,\, y\in\overline{M}} s \left(\sum_{\substack{\xi\in\mathcal{I} \\ |\partial_y^\beta \sigma_{A,L}(y,\xi)|\geq s}} \max\{\|u_\xi\|^2_{L^\infty(\overline{M})}, \|v_\xi\|^2_{L^\infty(\overline{M})}\} \right)^{\frac{1}{p}-\frac{1}{q}} < \infty,$$
for all $|\beta| \leq \left[\frac{\dim(M)}{q}\right] + 1$, where ∂_y denotes the local partial derivative. If $\partial M \neq \emptyset$, let us assume additionally that $\mathrm{supp}(\sigma_{A,L}) \subset \{(y,\xi) \in \overline{M} \times \mathcal{I} : y \in \overline{M} \setminus V\}$ where $V \subset \overline{M}$ is an open neighbourhood of the boundary ∂M. Then A admits a bounded extension from $L^p(\overline{M})$ into $L^q(\overline{M})$.

Applications to non-linear equations

We apply our L^p-L^q boundedness to establish the well–posedness properties the solutions of nonlinear equations in the space $L^\infty(0, T; L^2(M))$.

- In the nonlinear stationary problem case, we consider the following equation in $L^2(M)$
$$Au = |Bu|^p + f,$$
where $A, B : L^2(M) \to L^2(M)$ are linear operators and $1 \leq p < \infty$.
- In the case of the nonlinear heat equation, we study the Cauchy problem in the space $L^\infty(0, T; L^2(M))$
$$u_t(t) - |Bu(t)|^p = 0, u(0) = u_0,$$
where B is a linear operator in $L^2(M)$ and $1 \leq p < \infty$.
- In the non-linear wave equation case, we study the following initial value problem (IVP)
$$u_{tt}(t) - b(t)|Bu(t)|^p = 0,$$
$$u(0) = u_0, \quad u_t(0) = u_1,$$
where b is a positive bounded function depending only on time, B is a linear operator in $L^2(M)$ and $1 \leq p < \infty$.

References

[1] Akylzhanov, R., Ruzhansky, M., Nursultanov, E. Hardy-Littlewood, Hausdorff-Young-Paley inequalities, and L^p-L^q Fourier multipliers on compact homogeneous manifolds. *J. Math. Anal. Appl.* 479 (2019), no. 2, 1519–1548.

[2] Akylzhanov, R., Ruzhansky, M. L^p-L^q multipliers on locally compact groups, *J. Func. Analysis*, 278(3) (2019), DOI: https://doi.org/10.1016/j.jfa.2019.108324

[3] D. Cardona, V. Kumar, M. Ruzhansky and N. Tokmagambetov, L^p-L^q boundedness of pseudo-differential operators on smooth manifolds and its applications to nonlinear equations, (2020). *https://arxiv.org/abs/2005.04936*

[4] Delgado, J., Ruzhansky, M., Tokmagambetov, N. Schatten classes, nuclearity and nonharmonic analysis on compact manifolds with boundary, *J. Math. Pures Appl.*, 107:758–783, 2017.

[5] Hörmander, L. Estimates for translation invariant operators in L^p spaces. *Acta Math.*, 104, (1960) 93–140.

[6] Ruzhansky M., Tokmagambetov N., Nonharmonic analysis of boundary value problems, *Int. Math. Res. Notices*, (2016) 2016 (12), 3548–3615.

[7] Ruzhansky, M., Turunen, V. *Pseudo-differential Operators and Symmetries: Background Analysis and Advanced Topics.* Birkhaüser-Verlag, Basel, (2010).

[8] Taylor M., *Pseudodifferential Operators.* Princeton Univ. Press, Princeton, 1981.

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