Abstract

L-pseudo-differential operators on manifolds with boundary

- An L-pseudo-differential operator is a continuous linear operator $A : C^0_c(\mathcal{M}) \to C^0_c(\mathcal{M})$, defined by
  $$A(f) = \sum_{\xi \in \mathbb{R}^n} \langle \xi \rangle^{s} a(x,\xi)f(x),$$
  where $s$ is the order of $A$.

- In this case, the function $m: \mathcal{M} \times \mathbb{T} \to \mathbb{C}$, called the L-symbol associated with $A$.
- Denote by $L^p$ the densely defined operator given by $L^p(u) = \sum_{\xi \in \mathbb{R}^n} \langle \xi \rangle^{s} a(x,\xi)u(x)$.
- An L-pseudo-differential operator $A$ is called an L-multiplicator if there exists a continuous function $\tau_m : \mathcal{M} \times \mathbb{R} \to \mathbb{C}$ such that for every $\xi \in \mathbb{R}^n$ and $\epsilon \in \mathbb{R}$, we have $m(\xi,\epsilon) = \tau_m(x,\xi)$. So, $A$ is given by
  $$A(f) = \sum_{\xi \in \mathbb{R}^n} \langle \xi \rangle^{s} \tau_m(x,\xi)f(x),$$
  for all $f \in C^0_c(\mathcal{M})$.

Assumptions for H-M theorem

- There exist $\alpha < \gamma$, $\gamma_1 < \beta$, satisfying $|\tilde{u}(\xi)|, |\tilde{v}(\xi)| \leq |\xi|^\alpha$, for $1 \leq p \leq \infty$.
- The operator $A(L)$ satisfies the Weyl-epsilon counting formula

$$N(\lambda) = \sum_{\xi \in \mathbb{R}^n} O(\lambda^\delta), \lambda \to \infty,$$

where $Q > 0$. If $Q > Q$, then $N(\lambda) = O(\lambda^\delta), \lambda \to \infty$, so that we assume that $Q$ is the smallest real number with this property.

Theorem (Hörmander-Mihlin (H-M) theorem for pseudo-multipliers on $\mathcal{M}$)

Let $\mathcal{M}$ be a smooth manifold with boundary and let $A : C^0_c(\mathcal{M}) \to C^0_c(\mathcal{M})$ be an L-pseudo-multiplier. Let us assume that $\tau_m$ satisfies the following H-M condition,

$$||\tau_m(x,\xi)||_{L^p(\mathcal{M})} \leq \sum_{\xi \in \mathbb{R}^n} \langle \xi \rangle^{s} \tau_m(x,\xi)f(x),$$

for all $f \in C^0_c(\mathcal{M})$.

Then $A \in L^p(\mathcal{M})$, extends to a bounded linear operator on all $1 < p < \infty$.

Applications to non-linear equations

We apply our $L^p$-boundedness to establish the well-posedness properties of the solutions of nonlinear equations in the space $L^p(\mathcal{M})$. We consider the nonlinear stationary problem:

$$Au = f,$$

where $A : L^p(\mathcal{M}) \to L^p(\mathcal{M})$ are linear operators and $1 \leq p < \infty$.

In the case of the nonlinear heat equation, we study the Cauchy problem in the space $L^p(\mathcal{M})$.

References


