Hardy-Littlewood inequality and $L^p-L^q$ Fourier multipliers on compact hypergroups

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**Aim**

- To establish Hörmander’s $L^p-L^q$ Fourier multiplier theorem on compact hypergroups for the range $1 < p \leq 2 \leq q < \infty$ $[8]$.

**Methods**

- An application of Paley inequality gives Hardy-Littlewood inequality for compact hypergroups.

**H-L inequality for Conj(SU(2))**

If $1 < p \leq 2$ and $f \in L^p(\text{Conj}(SU(2)))$, then we have

$$\sum_{n \in \mathbb{N}} (2^j + 1)^{\frac{p}{q}} |\hat{f}(n)|^q \leq C(\|f\|_{L^p(\text{Conj}(SU(2)))})^q.$$


If $1 < p \leq 2$ then there exists a constant $C = C(p)$ such that

$$f(0) + \sum_{\tau \in \mathbb{Z}} (1 - \delta_{0, \tau}) |\hat{f}(\tau)|^p \leq C(\|f\|_{L^p(K)})^p.$$

**Hypergroups: What & why?**

- Roughly, a hypergroup $K$ is a locally compact Hausdorff space with a convolution on the space $M_b(K)$ of regular bounded Borel measures on $K$ with properties similar to those of group convolution.
- In non commutative setting, the analysis on hypergroups provides a natural extension of analysis on locally compact groups. While in commutative setting, they extend the theory of spherical functions and Gelfand pairs.
- Some of important examples are double coset spaces, the space of conjugacy classes on (Lie) groups and the space of group orbits.
- In particular, the results presented here are true for several interesting examples including Jacobi hypergroups with Jacobi polynomials as characters, compact hypergroup structure on the fundamental alcove with Heckman-Opdam polynomials as characters and multivarient disk hypergroups.
- A compact hypergroup can be countable infinite also $[6]$. This property distinguishes them from compact groups.
- Unlike locally compact abelian groups, the support of the Plancherel measure on the dual space may not be full space in the case of commutative hypergroups.
- For more details on analysis on hypergroups and several interesting examples, see $[4, 10, 6]$.

**Hausdorff-Young-Paley inequality on compact hypergroups**

**Theorem (Hausdorff-Young-Paley (Pitt) inequality)**

Let $K$ be a compact hypergroup and let $1 < p \leq b \leq b' < \infty$. If a positive sequence $\varphi(\pi), \pi \in K$, satisfies the condition

$$M_p := \sup_{\pi \in K} \sum_{\tau \in \mathbb{Z}} \|\varphi(\pi)\|^p \tau^p < \infty,$$

then we have:

$$\left( \sum_{\tau \in \mathbb{Z}} \left( \frac{\|\varphi(\pi)\|_1}{\sqrt{\lambda(\tau)}} \right)^p \tau^p \right)^{\frac{1}{p}} \leq M_{b'}^{-\frac{1}{b'}} \|f\|_{L^b(K)}.$$

**Non-commutative version of Hardy-Littlewood inequality**

**Theorem (Hardy-Littlewood inequality for compact hypergroups)**

Let $1 < p \leq 2$ and let $K$ be a compact hypergroup. Assume that a sequence $\{\mu_{\pi}\}_{\pi \in K}$ grows sufficiently fast, that is,

$$\sum_{\pi \in K} \frac{k_{\pi}^2}{|\mu_{\pi}|} \beta < \infty \quad \text{for some } \beta \geq 0.$$

Then we have:

$$\sum_{\pi \in K} \frac{k_{\pi}^2}{|\mu_{\pi}|} \beta (\frac{\|\varphi(\pi)\|_1}{\sqrt{\lambda(\tau)}})^p \tau^p \leq \|f\|_{L^b(K)}.$$

**References**


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