Hardy-Littlewood inequality and L^p-L^q Fourier multipliers on compact hypergroups

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Aim

- To prove Hardy-Littlewood inequality and Paley inequality for compact hypergroups [8].
- To establish Hörmander's L^p-L^q Fourier multiplier theorem on compact hypergroups for the range $1 < p \leq 2 \leq q < \infty$ [8].

Hypergroups: What & why?

- Roughly, a hypergroup K is a locally compact Hausdorff space with a convolution on the space $M_b(K)$ of regular bounded Borel measures on K with properties similar to those of group convolution.
- In non commutative setting, the analysis on hypergroups provides a natural extension of analysis on locally compact groups. While in commutative setting, they extend the theory of spherical functions and Gelfand pairs.
- Some of important examples are double coset spaces, the space of conjugacy classes on (Lie) groups and the space of group orbits.
- In particular, the results presented here are true for several interesting examples including Jacobi hypergroups with Jacobi polynomials as characters, compact hypergroup structure on the fundamental alcove with Heckman-Opdam polynomials as characters and multivariant disk hypergroups.
- A compact hypergroup can be countable infinite also ([6]). This property distinguishes them from compact groups.
- Unlike locally compact abelian groups, the support of the Plancherel measure on the dual space may not be full space in the case of commutative hypergroups.
- For more details on analysis on hypergroups and several interesting examples, see [4, 10, 6].

Fourier analysis on compact hypergroups

- Let K be compact hypergroup with normalised Haar measure λ. Denote by $\widehat{K}$ the dual space consisting of irreducible inequivalent continuous representations of K equipped with the discrete topology.
- Every $\pi \in \widehat{K}$ is finite dimensional but may not be unitary in contrast to compact groups case.
- In commutative setting also, the dual space $\widehat{K}$ may not have a hypergroup structure, in contrary to abelian groups.
- Denote the dimension and hyperdimension of $\pi \in \widehat{K}$ by d_π and k_π.
- For each $\pi \in \widehat{K}$, the Fourier transform $\widehat{f}$ of $f \in L^1(K)$ is defined as

$$\widehat{f}(\pi) = \int_K f(x)\bar{\pi}(x)\, d\lambda(x),$$

  where $\bar{\pi}$ is the conjugate representation of π.
- We refer to [10, 4, 9] for more details on Fourier analysis and representation theory of compact hypergroups.

Methods

- To establish Hausdorff-Young-Paley inequality we first prove Paley inequality [2, 11] for compact hypergroups [8] and then we use weighted interpolation with Hausdorff-Young inequality [9].
- An application of Paley inequality gives Hardy-Littlewood inequality for compact hypergroups.
- We obtain Hörmander L^p-L^q boundedness of Fourier multiplier [7] in context of compact hypergroup with the help of the Hausdorff-Young-Paley inequality.

Literature

- Hardy-Littlewood inequality [5] was recently estalished for compact homogeneous spaces [2] and for compact quantum groups [1, 11].
- L^p-L^q boundedness of Fourier multipliers on locally compact unimodular groups was proved in [3] using von-Neumann algebra techniques.

Hausdorff-Young-Paley inequality on compact hypergroups

Theorem (Hausdorff-Young-Paley (Pitt) inequality)

Let K be a compact hypergroup and let $1 < p \leq b \leq p' < \infty$. If a positive sequence $\varphi(\pi), \pi \in \widehat{K}$, satisfies the condition

$$M_\varphi := \sup_{y>0} y \sum_{\substack{\pi \in \widehat{K} \\ \varphi(\pi) \geq y}} k_\pi^2 < \infty,$$

then we have

$$\left(\sum_{\pi \in \widehat{K}} k_\pi^2 \left(\frac{\|\widehat{f}(\pi)\|_{\mathrm{HS}}}{\sqrt{k_\pi}} \varphi(\pi)^{\frac{1}{b} - \frac{1}{p'}} \right)^b \right)^{\frac{1}{b}} \lesssim M_\varphi^{\frac{1}{b} - \frac{1}{p'}} \|f\|_{L^p(K)}.$$

Non-commutative version of Hardy-Littlewood inequality

Theorem (Hardy-Littlewood inequality for compact hypergroups)

Let $1 < p \leq 2$ and let K be a compact hypergroup. Assume that a sequence $\{\mu_\pi\}_{\pi \in \widehat{K}}$ grows sufficiently fast, that is,

$$\sum_{\pi \in \widehat{K}} \frac{k_\pi^2}{|\mu_\pi|^\beta} < \infty \quad \textit{for some } \beta \geq 0.$$

Then we have

$$\sum_{\pi \in \widehat{K}} k_\pi^2 |\mu_\pi|^{\beta(p-2)} \left(\frac{\|\widehat{f}(\pi)\|_{\mathrm{HS}}}{\sqrt{k_\pi}} \right)^p \lesssim \|f\|_{L^p(K)}.$$

L^p-L^q-boundedness of Fourier multipliers on compact hypergroups

Theorem (Hörmander theorem for Fourier multipliers)

Let K be a compact hypergroup and let $1 < p \leq 2 \leq q < \infty$. Let A be a left Fourier multiplier with symbol σ_A, that is, A satisfies

$$\widehat{Af}(\pi) = \sigma_A(\pi)\widehat{f}(\pi), \quad \pi \in \widehat{K}.$$

Then we have

$$\|A\|_{L^p(K) \to L^q(K)} \lesssim \sup_{y>0} y \left(\sum_{\substack{\pi \in \widehat{K} \\ \|\sigma_A(\pi)\|_{op} \geq y}} k_\pi^2 \right)^{\frac{1}{p} - \frac{1}{q}}.$$

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H-L inequality for Conj(SU(2))

If $1 < p \leq 2$ and $f \in L^p(\mathrm{Conj}(\mathrm{SU})(2))$, then we have

$$\sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{5p-8} |\widehat{f}(l)|^p \leq C \|f\|_{L^p(\mathrm{Conj}(\mathrm{SU})(2))}$$

H-L for D-R hypergroups [6]

If $1 < p \leq 2$ then there exists a constant $C = C(p)$ such that

$$f(0) + \sum_{n \in \mathbb{N}} ((1-a)a^{-n})^{p(\frac{5}{2} - \frac{4}{p})} |\widehat{f}(n)|^p \leq C \|f\|_{L^p(H_a)}$$

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