Van der Corput lemmas for Mittag-Leffler functions

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Introduction

In harmonic analysis, one of the most important estimates is the van der Corput lemma, which is an estimate of the oscillatory integrals. This estimate was first obtained by the Dutch mathematician Johannes Gaultherus van der Corput [1] and named in his honour. While the paper [1] was published in Mathematische Annalen in 1921, he submitted it there on 17 December 1920 (from Utrecht). Therefore, it seems appropriate to us to dedicate this paper to the 100^{th} anniversary of this lemma.
Let us state the classical van der Corput lemmas as follows:

- **van der Corput lemma.** Suppose ϕ is a real-valued and smooth function in $[a,b]$. If ψ is a smooth function and $|\phi^{(k)}(x)| \geq 1$, $k \geq 1$, for all $x \in (a,b)$, then
$$\left|\int_a^b \exp(i\lambda\phi(x))\psi(x)dx\right| \leq \frac{C}{\lambda^{1/k}}, \quad \lambda \to \infty, \qquad (1)$$
for $k=1$ and ϕ' is monotonic, or $k \geq 2$. Here C does not depend on λ.

Formulation of problem

The main goal of the present paper is to study van der Corput lemmas for the oscillatory integrals defined by (see [2], [3]) respectively:
$$I_{\alpha,\beta}(\lambda) = \int_{\mathbb{R}} E_{\alpha,\beta}(i\lambda\phi(x))\,\psi(x)dx, \qquad (2)$$
and
$$\mathcal{I}_{\alpha,\beta}(\lambda) = \int_{\mathbb{R}} E_{\alpha,\beta}(i^\alpha\lambda\phi(x))\,\psi(x)dx, \qquad (3)$$
where $\alpha > 0$, $\beta > 0$, ϕ is a phase and ψ is an amplitude, and λ is a positive real number that can vary. Here $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function defined as
$$E_{\alpha,\beta}(z) = \Sigma_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha > 0, \quad \beta \in \mathbb{R},$$
with the property that
$$E_{1,1}(z) = e^z. \qquad (4)$$

Main results for $I_{\alpha,\beta}$

van der Corput lemmas on $\mathbb{R}$: consider $I_{\alpha,\beta}$ defined by (2).
Let $\phi : \mathbb{R} \to \mathbb{R}$ be a measurable function and let $\psi \in L^1(\mathbb{R})$. Suppose that $0 < \alpha < 1$, $\beta > 0$, and $m = \operatorname{ess\,inf}_{x\in\mathbb{R}} |\phi(x)| > 0$, then
$$|I_{\alpha,\beta}(\lambda)| \leq \frac{M}{1+\lambda m}\|\psi\|_{L^1(\mathbb{R})}, \ \lambda \geq 1,$$
where M does not depend on ϕ, ψ and λ.

van der Corput lemmas on $I = [a,b] \subset \mathbb{R}$, $-\infty < a < b < +\infty$.

- Let $0 < \alpha < 1$, $\beta > 0$, ϕ be a real-valued function such that $\phi \in C^k(I)$, $k \geq 1$, and let $\psi \in C^1(I)$. If $|\phi^{(k)}(x)| \geq 1$ for all $x \in I$, then
$$|I_{\alpha,\beta}(\lambda)| \leq M_k \lambda^{-\frac{1}{k}} \log^{\frac{1}{k}}(1+\lambda), \quad \lambda \geq 1,$$
where M_k does not depend on λ.
- Let $-\infty < a < b < +\infty$ and $I = [a,b] \subset \mathbb{R}$. Let $0 < \alpha \leq 1$ and let ϕ be a real-valued function such that $\phi \in C^k(I)$, $k \geq 1$. Let $\psi \in C^1(I)$ and $|\phi^{(k)}(x)| \geq 1$ for all $x \in I$. Then
$$|I_{\alpha,\alpha}(\lambda)| \leq M_k \lambda^{-1/k}, \quad \lambda \geq 1, \qquad (5)$$
for $k = 1$ and ϕ' is monotonic, or $k \geq 2$. Here M_k does not depend on λ.

Main results for $\mathcal{I}_{\alpha,\beta}$

van der Corput lemmas on $\mathbb{R}$: consider $\mathcal{I}_{\alpha,\beta}$ defined by (3).
Let $\phi : \mathbb{R} \to \mathbb{R}$ be a measurable function and let $\psi \in L^1(\mathbb{R})$. Suppose that $0 < \alpha \leq 2$, $\beta > 1$, and $m = \operatorname{ess\,inf}_{x\in\mathbb{R}} |\phi(x)| > 0$, then
$$|\mathcal{I}_{\alpha,\beta}(\lambda)| \leq \frac{M}{1+\lambda m}\|\psi\|_{L^1(\mathbb{R})}, \ \lambda \geq 1, \ 0 < \alpha < 2, \ \beta \geq \alpha+1,$$
$$|\mathcal{I}_{\alpha,\beta}(\lambda)| \leq \frac{M}{(1+\lambda m)^{\frac{\beta-1}{\alpha}}}\|\psi\|_{L^1(\mathbb{R})}, \ \lambda \geq 1, \ 0 < \alpha \leq 2, \ 1 < \beta < \alpha+1,$$
where M does not depend on ϕ, ψ and λ.

van der Corput lemmas on $I = [a,b] \subset \mathbb{R}$, $-\infty < a < b < +\infty$.

- Let $0 < \alpha < 2$, $\beta > 1$, ϕ be a real-valued function such that $\phi \in C^k(I)$, $k \geq 1$, and let $\psi \in C^1(I)$. If $|\phi^{(k)}(x)| \geq 1$ for all $x \in I$, then
$$|\mathcal{I}_{\alpha,\beta}(\lambda)| \leq M_k \lambda^{-\frac{1}{k}} \log^{\frac{1}{k}}(1+\lambda), \quad \lambda \geq 1, \ 0 < \alpha < 2, \ \beta \geq \alpha+1,$$
$$|\mathcal{I}_{\alpha,\beta}(\lambda)| \leq M_k \lambda^{-\frac{1}{k}} (1+\lambda)^{\frac{\alpha+1-\beta}{\alpha k}}, \quad \lambda \geq 1, \ 0 < \alpha < 2, \ 1 < \beta < \alpha+1,$$
where M_k does not depend on λ.
- Let $-\infty < a < b < +\infty$ and $I = [a,b] \subset \mathbb{R}$. Let $0 < \alpha < 2$ and let ϕ be a real-valued function such that $\phi \in C^k(I)$, $k \geq 1$. Let $\psi \in C^1(I)$ and $|\phi^{(k)}(x)| \geq 1$ for all $x \in I$. Then
$$|\mathcal{I}_{\alpha,\alpha}(\lambda)| \leq M_k \lambda^{-1/k}, \quad \lambda \geq 1, \qquad (6)$$
for $k = 1$ and ϕ' is monotonic, or $k \geq 2$. Here M_k does not depend on λ.

Remark

The case of $\alpha = 1$ in (5) and (6) corresponds to the classical van der Corput lemma (1).

Conclusion

The main goal of the paper was to study van der Corput lemmas for the integrals defined by (2) and (3). Van der Corput type lemmas were obtained, for the different cases of parameters α and β.
As an immediate application of the obtained results, time-estimates of the solutions of time-fractional Klein-Gordon and Schrödinger equations and generalisations of the Riemann-Lebesgue lemma were also considered in [2] and [3].

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