Van der Corput lemmas for Mittag-Leffler functions

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Introduction

In harmonic analysis, one of the most important estimates is the van der Corput lemma, which is an estimate of the oscillatory integrals. This estimate was first obtained by the Dutch mathematician Johannes Hadamard van der Corput in 1915 and named in his honour. While the paper [1] was published in Mathematische Annalen in 1921, it submitted it there on 17 December 1920 (from Utrecht). Therefore, it seems appropriate to us to dedicate this paper to the 100th anniversary of this lemma.

Let us state the classical van der Corput lemma as follows:

**van der Corput lemma.** Suppose $\phi$ is a real-valued and smooth function in $[a, b]$. If $\psi$ is a smooth function and $|\phi'(z)| \geq 1$, $k \geq 1$, for all $x \in (a, b)$, then

$$\int_a^b \exp (i \phi(x)) \psi(x) dx \leq \frac{C}{\lambda^k}, \quad \lambda \to \infty,$$

for $k = 1$ and $\phi'$ is monotonic, or $k \geq 2$. Here $C$ does not depend on $\lambda$.

Formulation of problem

The main goal of the present paper is to study van der Corput type lemmas for Mittag-Leffler functions. Van der Corput lemmas for Mittag-Leffler functions.

van der Corput lemmas on $\mathbb{R}$: consider $I_{a, \beta}$ defined by (2).

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a measurable function and let $\psi \in L^1(\mathbb{R})$. Suppose that $0 < \alpha < 1$, $\beta > 0$, and $m = \text{ess inf}_{x \in \mathbb{R}} |\phi(x)| > 0$, then

$$|I_{a, \beta}(\lambda)| \leq \frac{M}{1 + \lambda m} \|\psi\|_{L^1(\mathbb{R})}, \quad \lambda \geq 1,$$

where $M$ does not depend on $\phi$, $\psi$ and $\lambda$.

van der Corput lemmas on $I = [a, b] \subset \mathbb{R}_{-\infty} < a < b < +\infty$.

- Let $0 < \alpha < 1$, $\beta > 0$, $\phi$ be a real-valued function such that $\phi \in C^k(I)$, $k \geq 1$, and let $\psi \in C^1(I)$. If

$$|\phi'(z)| \geq 1 \text{ for all } z \in I,$$

then

$$|I_{a, \beta}(\lambda)| \leq M_0 \lambda^{-\frac{1}{k}}, \quad \lambda \geq 1,$$

for $k = 1$ and $\phi'$ is monotonic, or $k \geq 2$. Here $M_0$ does not depend on $\lambda$.

Main results for $I_{a, \beta}$

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Remark

The case of $\alpha = 1$ in (5) and (6) corresponds to the classical van der Corput lemma (1).

Conclusion

The main goal of the paper was to study van der Corput lemmas for the integrals defined by (2) and (3). Van der Corput type lemmas were obtained, for the different cases of parameters $\alpha$ and $\beta$.

As an immediate application of the obtained results, time-estimates of the solutions of time-fractional Klein-Gordon and Schrödinger equations and generalizations of the Riemann-Lebesgue lemma were also considered in [2] and [3].

Acknowledgements

The authors were supported in parts by the FWO Odysseus 1 grant G.0894.18N: Analysis and Partial Differential Equations. The first author was also supported by EPSRC grant EP/R003025/1 and by the Leverhulme Grant RPG-2017-151.

References