Some inverse source problems in semilinear fractional PDEs

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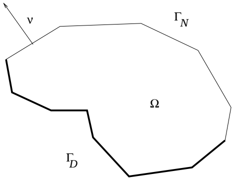
Outline

1. Inverse source problems
2. Fractional calculus
3. ISP for time-fractional parabolic equation
4. Time-fractional hyperbolic ISP

Introduction



Example: A general linear parabolic PDE with mixed BCs

$$
\begin{aligned}
u_t + \nabla \cdot (-\boldsymbol{A}_{dif} \nabla u - \boldsymbol{a}_{con} u) + a_{sou} u &= f + \nabla \cdot \boldsymbol{f}_{div} && \text{in} \quad \Omega \\
u &= g_{Dir} && \text{on} \quad \Gamma_D \\
(-\boldsymbol{A}_{dif} \nabla u - \boldsymbol{a}_{con} u)^T \boldsymbol{\nu} - g_{Rob} u &= g_{Neu} && \text{on} \quad \Gamma_N \\
u(x,0) &= u_0(x) && \text{in} \quad \Omega
\end{aligned}
$$

Game of IPs

- What is known/unknown?
- Additional data
- Well-posedness

ISPs in the literature

Consider a PDE of the type $Lu(t,x) = f(t,x)$.
Literature research gives many of papers devoted to the recovery of

- $f(x)$ based on e.g. on final measurement [Rundell, 1980, Cannon, 1968, Prilepko and Solov'ev, 1988, Solov'ev, 1990, Isakov, 1990, Farcas and Lesnic, 2006, Hasanov, 2007, Johansson and Lesnic, 2007a, Johansson and Lesnic, 2007b]...
- $f(t)$ based on local or non-local measurement. [Prilepko et al., 2000, Hasanov, 2011, Yang et al., 2011, Hasanov and Slodička, 2013, Slodička, 2013, Hazanee et al., 2013, Hasanov, 2011, Hasanov and Pektaş, 2013]...

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Example spectral analysis . . .

Let us consider the following homogeneous problem in $\Omega = (0, 1)$

$$\begin{aligned} -u''(x) &= f(x) \qquad x \in \Omega, \\ u(0) = u(1) &= 0. \end{aligned} \tag{1}$$

We denote by $A: \ D(A) \to X$ the second order differential operator, where

$$A = -\frac{d^2}{dx^2}, \qquad D(A) = H^2(\Omega) \cap H_0^1(\Omega)$$

and $X = L^2(\Omega)$. The spectrum $\sigma(A)$ of the operator A consists of the eigenvalues

$$\lambda_n = \pi^2 n^2, \qquad n \in \mathbb{N}.$$

The corresponding eigenfunctions have the form

$$e_n(x) = \sqrt{2}\ \sin(n\pi x), \qquad n \in \mathbb{N}.$$

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. . . Example spectral analysis

The set of all eigenfunctions is an orthonormal complete system in $L^2(\Omega)$. Thus

$$(e_n, e_m) = \int_\Omega e_n(x)e_m(x)\,\mathrm{d}x = \delta_{n,m}{}^1.$$

Each function $f \in L^2(\Omega)$ can be written as

$$f = \sum_{i=1}^{\infty}(f, e_i)e_i,$$

where $(\cdot,\cdot)$ is the inner product in X

$$(f, e_i) = \int_\Omega f(x)e_i(x)\,\mathrm{d}x.$$

[1] This is the Kronecker symbol $\delta_{n,m} = 1$ if $n = m$, else $\delta_{n,m} = 0$.

Example ISP . . .

IS Problem: Find $(u(t,x), h(t))$ such that

$$\begin{aligned} u_t - u''(x) &= h(t)f(x) \qquad x \in \Omega = (0,1), \\ u(0) = u(1) &= 0 \\ u(0,x) &= 0 \\ u(t,x_0) &= m(t) \qquad x_0 \in \Omega \end{aligned} \tag{2}$$

Question: Is the solution unique?
Answer: If yes, then $m(t) = 0 \implies (u(t,x), h(t)) = (0,0)$.

. . . Example ISP . . .

Take an eigenfunction w of the operator $Au = -u''$, i.e. $Aw = \lambda w$. Set $f = w$. Seek the solution u in the form $u(t,x) = \alpha(t)w(x)$. From the PDE we get

$$[\alpha'(t) + \lambda\alpha(t)]\, w(x) = h(t)w(x)$$

Thus α solves

$$\alpha'(t) + \lambda\alpha(t) = h(t); \qquad \alpha(0) = 0.$$

Therefore

$$\alpha(t) = \int_0^t e^{-\lambda(t-s)} h(s)\, ds.$$

Non uniqueness: If x_0 is a zero point of $w(x)$, i.e. $w(x_0) = 0$, which implies $m(t) = 0$, but we have at least 2 solutions

$$(u,h) = (0,0), \qquad (u,h) = \left(w(x) \int_0^t e^{-\lambda(t-s)} h(s)\, ds, h(t) \right).$$

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. . . Example . . .

Bad choice of a measurement point: Zero point of an eigenfunction.
How many eigenfunctions do I have?: ∞
Recall that

$$e_n(x) = \sqrt{2}\,\sin(n\pi x), \qquad n \in \mathbb{N}.$$

What are all zero poins of all eigenfunctions?
Solving

$$0 = \sin(n\pi x) \Longrightarrow n\pi x = m\pi \qquad m \in \mathbb{Z}$$

we get

$$x = \frac{m}{n}.$$

Zero points of all eigenfunctions are dense in Ω.

. . . Example

Which condition ensures the uniqueness for the ISP? A nonlocal one

$$(u(t),1)_\omega := \int_\omega u(t,x)\,\mathrm{d}x = m(t) \qquad \overline{\omega} \subset \Omega, \quad f \in C_0^1(\Omega).$$

Multiply PDE by 1 and integrate over ω to see

$$(\partial_t u(t),1)_\omega - (u''(t),1)_\omega = h(t)(f,1)_\omega \Longrightarrow h(t) = \ldots$$

Multiply PDE by $-u''$ and integrate over Ω to see ($m(t) = 0$ by uniqueness)

$$\tfrac{1}{2}\partial_t \left\|u'(t)\right\|^2 + \left\|u''(t)\right\|^2 = \frac{(u''(t),1)_\omega}{(f,1)_\omega}(f,u'') = \frac{-(u''(t),1)_\omega}{(f,1)_\omega}(f',u')$$

Young's inequality

$$\tfrac{1}{2}\partial_t \left\|u'(t)\right\|^2 + (1-\varepsilon)\left\|u''(t)\right\|^2 \le C_\varepsilon \left\|u'(t)\right\|^2.$$

Grönwall's lemma implies the uniqueness.

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Rothe's method as a semigroup

Example 1 ($D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, $t \in [0,T]$, $u(0) = u_0$)

Idea for $x \in \mathbb{R}$: $e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$. Set $\tau = \frac{T}{n}$.

continue problem	Rothe's method
$\partial_t u + Au = f(u)$	$\delta u_i + Au_i = \frac{u_i - u_{i-1}}{\tau} + Au_i = f(u_{i-1})$
$S(t) := e^{-At}$	$S_\tau(t) := (I + \tau A)^{\frac{t}{\tau}}$
$u(t) = S(t)u_0 + \int_0^t S(t-s)f(u(s))\,\mathrm{d}s$	$u_i = S_\tau(t_i)u_0 + \sum_{k=0}^{i-1} S_\tau(t_i - t_k)f(u_k)\tau$

- Semigroups $S(t) := e^{-At}$, $\quad S_\tau(t) := (I + \tau A)^{\frac{t}{\tau}}$
- Error $\|u_i - u(t_i)\| \le C(\|A^\beta u_0\|)\tau^{\min\{1,\beta\}}$

Fractional derivatives . . .

What do we need?

- $D^0 = I$, Aditivity $D^\alpha D^\beta = D^{\alpha+\beta}$, for any $\alpha, \beta \in \mathbb{R}_+$
- Restriction of the fractional operator to natural numbers coincides with the classical derivative

$$D^\alpha = \frac{d^\alpha}{dx^\alpha} \qquad \text{for } \alpha \in \mathbb{N}$$

Assume that $\alpha \geq 0, x > a$

Riemann-Liouville

$$\begin{aligned}(D_a^\alpha y)(x) &:= \left(D^n I_a^{n-\alpha} y\right)(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x \frac{y(t)}{(x-t)^{\alpha-n+1}}\, dt\end{aligned}$$

Caputo

$$\begin{aligned}\left({}^C D_a^\alpha y\right)(x) &:= \left(I_a^{n-\alpha} D^n y\right)(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{y^{(n)}(t)}{(x-t)^{\alpha-n+1}}\, dt\end{aligned}$$

. . . Fractional derivatives

Theorem 2 (Relationship)

Let $\alpha \geq 0$, $y \in AC^n([a,b])$. *Then*

$$\left({}^C D_a^\alpha y\right)(x) = \left(D_a^\alpha y\right)(x) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{\Gamma(k-\alpha+1)}(x-a)^{k-\alpha}.$$

Properties

- Aditivity $D^\alpha D^\beta = D^{\alpha+\beta}$, for any $\alpha, \beta \in \mathbb{R}_+$
- Caputo = Sturm-Liouville if

$$y^{(k)}(a) = 0 \qquad \text{for } k = 0, \ldots, n-1$$

How to get a priori estimates?

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Example of a convolution kernel

Assume $v \in C[0,\infty)$ and $T > 0$. It holds

$$\frac{d}{dt}\left[\int_0^t \mathbf{e}^{-(t-s)} v(s)\,\mathrm{ds}\right] = v(t) - \int_0^t e^{-(t-s)} v(s)\,\mathrm{ds}.$$

So

$$\frac{d}{dt}\left[\int_0^t e^{-(t-s)} v(s)\,\mathrm{ds}\right]^2 = 2\left[\int_0^t e^{-(t-s)} v(s)\,\mathrm{ds}\right]\left[v(t) - \int_0^t e^{-(t-s)} v(s)\,\mathrm{ds}\right].$$

Integration in time over $[0, T]$ gives

$$2\int_0^T v(t)\int_0^t e^{-(t-s)} v(s)\,\mathrm{ds}\,\mathrm{dt} = \left[\int_0^T e^{-(T-s)} v(s)\,\mathrm{ds}\right]^2$$

$$+\int_0^T \left[\int_0^t e^{-(t-s)} v(s)\,\mathrm{ds}\right]^2 \mathrm{d}t \geq \mathbf{0}.$$

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Positive kernels [Nohel and Shea, 1976]

Let $a(t) \in L^1_{loc}(0,\infty)$ is of **positive** type if

$$\int_0^T v(t) \int_0^t v(\xi) a(t-\xi) \mathrm{d}\xi \, \mathrm{d}t \geq 0$$

for any $v \in C[0,\infty)$ and any $T > 0$.

A real function $a(t)$ is **strongly positive** if there exists $\eta > 0$ such that $b(t) = a(t) - \eta e^{-t}$ is of a positive type.

Let $a(t) \in L^1_{loc}(0,\infty)$ be **not constant, ≥ 0, nonincreasing, convex** and such that $da'(t)$ is not a purely singular measure. Then $a(t)$ is strongly positive.

In particular, twice-differentiable $a(t)$ satisfying

$$(-1)^k a^{(k)}(t) \geq 0; \qquad 0 < t < \infty, \; k = 0, 1, 2; \; a' \not\equiv 0$$

are strongly positive.

Caputo derivative as a convolution

Define (**Riemann-Liouville kernel**)

$$g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}, \qquad t > 0,\ \beta > 0$$

which is strongly positive definite.

By $K * u$ we denote the usual convolution in time, namely

$$(K * u(\boldsymbol{x}))(t) = \int_0^t K(t-s)u(\boldsymbol{x}, s)\, \mathrm{d}s.$$

The Caputo fractional derivative can be also rewritten as a convolution with g_β

$$\partial_t^\alpha v(t) = \frac{\partial^\alpha v}{\partial t^\alpha} := \begin{cases} \left(g_{1-\alpha} * \partial_t v\right)(t), & \alpha \in (0,1) \\ \left(g_{2-\alpha} * \partial_{tt} v\right)(t), & \alpha \in (1,2) \\ \partial_t v(t), & \alpha = 1 \end{cases}$$

How to get a priori estimates?

Zacher's lemma

- Let H be a real Hilbert space with a scalar product $(\cdot,\cdot)_H$ with the corresponding norm $\|\cdot\|_H$.

[Zacher, 2010, Lemma 2.3.2], [Zacher, 2008, Zacher, 2013] proved the following identity

$$\begin{aligned}\left(\tfrac{d}{dt}(k*v)(t), v(t)\right)_H &= \frac{1}{2}\frac{d}{dt}\left(k*\|v\|_H^2\right)(t) + \tfrac{1}{2}k(t)\,\|v(t)\|_H^2\\ &+\frac{1}{2}\int_0^t [-k'(s)]\,\|v(t)-v(t-s)\|_H^2\,\mathrm{ds} \quad a.e.\ t\in(0,T),\end{aligned}$$

which is valid for any $k \in H^{1,1}([0,T])$ and each $v \in L^2([0,T],H)$.

The assumption $k \in H^{1,1}([0,T])$ is too strong. What now?

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Crucial lemma ...

[Slodička and Šišková, 2016] CAMWA

The following crucial lemma which will play a central role in the proofs.

Lemma 3

Let H be a real Hilbert space with a scalar product $(\cdot,\cdot)_H$ *and the corresponding norm* $\|\cdot\|_H$*. Assume* $T > 0$, $g \in L^1(0,T)$, $g' \in L^{1,loc}(0,T)$, $g' \leq 0$, $g \geq 0$*. If* $v : [0,T] \to H$ *such that* $v(0) \in H$, $v \in H^1((0,T),H)$ *then*

$$\begin{aligned}\int_0^\xi \left(\frac{d}{dt}(g*v)(t), v(t)\right)_H dt &\geq \tfrac{1}{2}\left(g*\|v\|_H^2\right)(\xi) + \tfrac{1}{2}\int_0^\xi g(t)\,\|v(t)\|_H^2\, dt \\ &\geq \frac{g(T)}{2}\int_0^\xi \|v(t)\|_H^2\, dt\end{aligned}$$

for any $\xi \in [0,T]$.

. . . Crucial lemma

Proof.

- Start from the Zacher's lemma and replace k by $g_n(s) := \min\{n, g(s)\}$.
- It holds

$$g_n'(s) \leq 0, \qquad g_n(s) \to g(s) \qquad a.e. \text{ in } [0, T].$$

- Integrate in time
- Pass to the limit for $n \to \infty$.

□

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Inverse source problem . . .

Differential operator $\Omega \subset \mathbb{R}^d$, $t \in [0, T]$

$$
\begin{aligned}
L(x,t)u &= \nabla \cdot (-\boldsymbol{A}(x,t)\nabla u - \boldsymbol{b}(x,t)u) + c(t)u, \\
\boldsymbol{A}(x,t) &= (a_{i,j}(x,t))_{i,j=1,\ldots,d}\,, \\
\boldsymbol{b}(x,t) &= (b_1(x,t),\ldots,b_d(x,t)).
\end{aligned}
$$

Governing PDE

$$
(g_{1-\beta} * \partial_t u(x))\,(t) + L(x,t)u(x,t) = h(t)f(x) + \int_0^t F(x,s,u(x,s))\,\mathrm{d}s, \tag{3}
$$

where $g_{1-\beta}$ denotes the Riemann-Liouville kernel

$$
g_{1-\beta}(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}, \qquad t > 0,\ 0 < \beta < 1
$$

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$$
\begin{aligned}
u(x,0) &= u_0(x), \quad x \in \Omega \\
(-\boldsymbol{A}(x,t)\nabla u(x,t) - \boldsymbol{b}(x,t)u(x,t)) \cdot \boldsymbol{\nu} &= g(x,t) \quad (x,t) \in \Gamma \times (0,T).
\end{aligned} \tag{4}
$$

. . . Inverse source problem

- Find $(u(x,t), h(t))$ obeying (3), (4).
- The unknown time-dependent function $h(t)$ will be determined from the following additional measurement

$$m(t) = \int_\Omega u(x,t)\,\mathrm{d}x = (u(t), 1)\,, \qquad t \in [0, T]. \tag{5}$$

Variational framework . . .

We associate a bilinear form $\mathcal{L}$ with the differential operator L as follows

$$(Lu, \varphi) = \mathcal{L}\,(u, \varphi) + (g, \varphi)_\Gamma\,, \qquad \forall \varphi \in H^1(\Omega),$$

i.e.

$$\mathcal{L}(t)\,(u(t), \varphi) = (\boldsymbol{A}(t)\nabla u(t) + \boldsymbol{b}(t)u(t), \nabla\varphi) + c(t)\,(u(t), \varphi)\,.$$

Throughout the paper we assume that

$$\begin{array}{lll} a_{i,j}, b_i : \overline{\Omega} \times [0, T] \to \mathbb{R}, & |a_{i,j}| + |b_i| \le C, & i, j = 1, \ldots, d, \\ 0 \le c(t) \le C, & \forall t \in [0, T], & \\ \mathcal{L}(t)\,(\varphi, \varphi) \ge C_0 \left\|\nabla\varphi\right\|^2, & \forall \varphi \in H^1(\Omega), & \forall t \in [0, T]. \end{array} \tag{6}$$

. . . Variational framework

Integrate (3) over Ω, apply Green's theorem and use (5) to get

$$(g_{1-\beta} * m')(t) + c(t)m(t) = h(t)(f, 1) - (g(t), 1)_\Gamma + \int_0^t (F(s, u(s)), 1)\, \mathrm{d}s. \qquad \text{(MP)}$$

Assuming that $(f, 1) \neq 0$ we have

$$h(t) = \frac{(g_{1-\beta} * m')(t) + c(t)m(t) + (g(t), 1)_\Gamma - \int_0^t (F(s, u(s)), 1)\, \mathrm{d}s}{(f, 1)}. \qquad (7)$$

The variational formulation of (3) and (4) reads as

$$\begin{aligned} &((g_{1-\beta} * \partial_t u)(t), \varphi) + \mathcal{L}(t)(u(t), \varphi) \\ &= h(t)(f, \varphi) + \left(\int_0^t F(s, u(s))\, \mathrm{d}s, \varphi \right) - (g(t), \varphi)_\Gamma \end{aligned} \qquad \text{(P)}$$

for any $\varphi \in H^1(\Omega)$, a.a. $t \in [0, T]$ and $u(0) = u_0$.

Uniqueness

Theorem 4

Let $f, u_0 \in L^2(\Omega)$, $\int_\Omega f \neq 0$, $m \in C^1([0,T])$, F be a global Lipschitz continuous function in all variables. Assume (6) and $g \in C([0,T],\Gamma)$.
Then there exists at most one solution (u,h) to the (P), (MP) obeying
$u \in C\left([0,T], L^2(\Omega)\right) \cap L^\infty\left((0,T), H^1(\Omega)\right)$ *with* $\partial_t u \in L^2\left((0,T), L^2(\Omega)\right)$, $h \in C([0,T])$.

Proof.

Suppose that (u_i, h_i) for $i = 1, 2$ solve (P), (MP).
Set $u = u_1 - u_2$ and $h = h_1 - h_2$.
Subtract the corresponding variational formulations from each other.
Set $\varphi = u(t)$ in variational formulation and integrate in time over $(0, \xi)$.
Use crucial lemma. □

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Time discretization, Discrete convolution

Equidistant time-partitioning of $[0, T]$ with a step $\tau = T/n$, for any $n \in \mathbb{N}$. Set $t_i = i\tau$ and denote

$$z_i = z(t_i), \qquad \delta z_i = \frac{z_i - z_{i-1}}{\tau}.$$

Let us define the discrete convolution in time as follows

$$(K * v)_i := \sum_{k=1}^{i} K_{i+1-k} v_k \tau.$$

Please note that this definition allows blow up of K at $t = 0$. An easy calculation yields

$$\delta\,(K * v)_i = \frac{(K * v)_i - (K * v)_{i-1}}{\tau} = K_1 v_i + \sum_{k=1}^{i-1} \delta K_{i+1-k} v_k \tau, \qquad i \geq 1 \tag{8}$$

as $(K * v)_0 := 0$. Similarly we may write

$$\delta\,(K * v)_i = K_i v_0 + \sum_{k=1}^{i} \delta v_k K_{i+1-k} \tau = K_i v_0 + (K * \delta v)_i\,, \qquad i \geq 1. \tag{9}$$

Discrete crucial lemma

The following technical lemma is a discrete analogy of Lemma 3. It plays a central role by establishing a priori estimates for u_i and h_i.

Lemma 5

Let $\{v_i\}_{i\in\mathbb{N}}$ and $\{K_i\}_{i\in\mathbb{N}}$ be sequences of real numbers. Assume that K decreases, i.e. $K_i \le K_{i-1}$ for any i. Then

$$2\delta\left(K * v\right)_i v_i \ge \delta\left(K * v^2\right)_i + K_i v_i^2, \qquad i \in \mathbb{N}.$$

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Discrete problem

Consider a system with unknowns (u_i, h_i) for $i = 1, \ldots, n$. At time t_i we approximate (P) by

$$\left((g_{1-\beta} * \delta u)_i, \varphi\right) + \mathcal{L}_i(u_i, \varphi) = h_i(f, \varphi) + \left(\sum_{k=1}^{i} F(t_k, u_{k-1})\tau, \varphi\right) - (g_i, \varphi)_\Gamma \qquad (\mathrm{DP}i)$$

and (MP) by

$$(g_{1-\beta} * m')_i + c_i m_i = h_i(f, 1) + \left(\sum_{k=1}^{i} F(t_k, u_{k-1})\tau, 1\right) - (g_i, 1)_\Gamma . \qquad (\mathrm{DMP}i)$$

- Considering u_{k-1} in the argument of F makes (DPi) linear in u_i.
- The decoupling of u_i and h_i has been achieved by considering u_{k-1} in (DMPi).
- For a given $i \in \{1, \ldots, n\}$ solve first (DMPi) and then (DPi). Then increase i to $i + 1$.

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Existence of (u_i, h_i)

Lemma 6

Let $f, u_0 \in L^2(\Omega)$, $\int_\Omega f \neq 0$, $m \in C^1([0,T])$, F be a global Lipschitz continuous function in all variables. Assume (6) and $g \in C([0,T],\Gamma)$. Then for each $i \in \{1,\dots,n\}$ there exists a unique couple $(u_i, h_i) \in H^1(\Omega) \times \mathbb{R}$ solving (DPi) and (DMPi).

Proof.

Resolving (DMPi) for h_i we get

$$h_i = \frac{(g_{1-\beta} * m')_i + c_i m_i + (g_i, 1)_\Gamma - \left(\sum_{k=1}^{i} F(t_k, u_{k-1})\tau, 1\right)}{(f, 1)} \in \mathbb{R}. \tag{10}$$

Use Lax-Milgram lemma for (DPi). □

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Stability analysis . . .

Introduce the following notation

$$\left(g_{1-\beta} * \|u\|^2\right)_j = \sum_{k=1}^{j} g_{1-\beta}(t_{j+1-k}) \|u_k\|^2 \tau.$$

Lemma 7

Let the assumptions of Lemma 6 be fulfilled. Then there exist positive constants C and τ_0 such that for any $0 < \tau < \tau_0$ we have

(i) $\max_{1\le j\le n} \left(g_{1-\beta} * \|u\|^2\right)_j + \sum_{i=1}^{n} g_{1-\beta}(t_i) \|u_i\|^2 \tau + \sum_{i=1}^{n} \|u_i\|^2_{H^1(\Omega)} \tau \le C,$

(ii) $\max_{1\le j\le n} |h_j| \le C.$

. . . Stability analysis . . .

Compatibility condition: Assume that the (3) is fulfilled at $t = 0$, i.e. (P) holds true for $t = 0$. Therefore we may also put $t = 0$ in (MP), which allows us to define h_0 as follows

$$h_0 = \frac{c_0 m_0 + (g_0, 1)_\Gamma}{(f, 1)}. \tag{11}$$

Lemma 8

Let the assumptions of Lemma 6 be fulfilled. Moreover assume (11), $u_0 \in H^1(\Omega)$, $g \in C^1([0,T],\Gamma)$, $m \in C^2([0,T])$, $\partial_t c \in L^\infty(0,T)$ and $\partial_t a_{i,j}, \partial_t b_i \in L^\infty(\Omega \times (0,T))$ for all $i, j = 1, \dots, d$. Then there exist positive constants C and τ_0 such that for any $0 < \tau < \tau_0$ we have

(i) $\displaystyle \max_{1\le j\le n} \left(g_{1-\beta} * \|\delta u\|^2 \right)_j + \sum_{i=1}^{n} g_{1-\beta}(t_i) \|\delta u_i\|^2 \tau + \sum_{i=1}^{n} \|\delta u_i\|^2_{H^1(\Omega)} \tau \le C,$

(ii) $|\delta h_i| \le C + C t_i^{-\beta}$ *for any* $i = 1, \dots, n$.

Existence of a solution

Theorem 9

Let $f \in L^2(\Omega)$, $u_0 \in H^1(\Omega)$, $\int_\Omega f \neq 0$, $m \in C^2([0,T])$, *and* $g \in C^1([0,T],\Gamma)$. *Suppose that F is a global Lipschitz continuous function in all variables. Assume (6), (11),* $\partial_t c \in L^\infty[0,T]$ *and* $\partial_t a_{i,j}, \partial_t b_i \in L^\infty(\overline{\Omega \times (0,T)})$ *for all* $i,j = 1,\ldots,d$.
Then there exists a solution (u,h) *to the (P), (MP) obeying* $u \in C\left([0,T], H^1(\Omega)\right)$ *with* $\partial_t u \in L^2\left((0,T), H^1(\Omega)\right)$, $h \in C([0,T])$.

Proof.

- Cauchy, Young inequalities; Lebesgue dominated theorem
- Convergence (functional analysis)

□

Noisy data Regularization $m \approx m_\varepsilon \in C^2$

Example ...

We consider problem (P)-(MP) for $\Omega = (0.5, 3)$, $T = 3$ and $\beta = 0.5$ with

$$\begin{aligned} \mathcal{L}(u, \varphi) &= (\nabla u, \nabla \varphi), \\ f(x) &= \sin x, \\ F(x, t, u) &= -4tu \exp\left(1 - \frac{u^2}{\sin^2 x}\right), \end{aligned}$$

along with the initial and boundary conditions

$$\begin{aligned} u_0(x) &= 2 \sin x, \\ g(0.5, t) &= (t^2 + 2) \cos \tfrac{1}{2}, \\ g(3, t) &= (t^2 + 2) \cos 3, \end{aligned}$$

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. . . Example . . .

where the time-dependent measurement is

$$m(t) = \left(\cos\frac{1}{2} - \cos 3\right)\left(t^2 + 2\right).$$

One can easily verify that functions

$$u(x,t) = \left(t^2 + 2\right)\sin x$$

and

$$h(t) = \frac{8}{3\sqrt{\pi}} t^{\frac{3}{2}} + t^2 - \exp\left(1 - \left(t^2 - 2\right)^2\right) + e^{-3} + 2$$

solve the given problem.

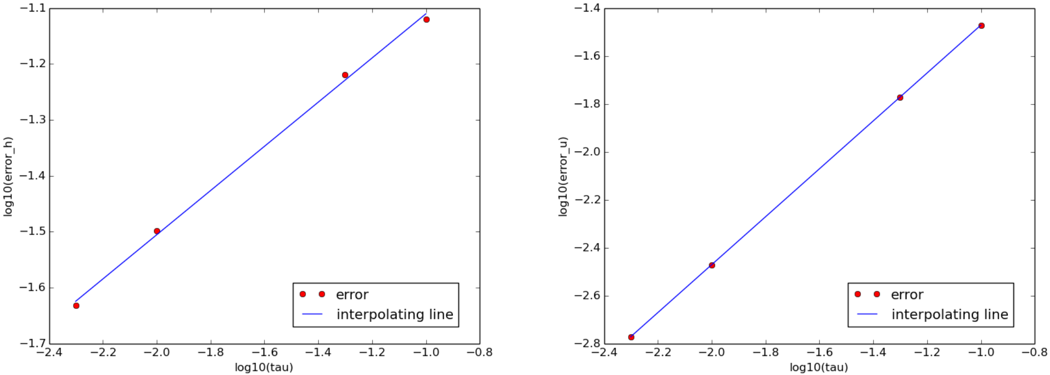
. . . Example . . .

Discretization parameters

- Ω is uniformly divided into 50 subintervals
- The solution u_i is calculated using a finite element method with Lagrange polynomials of the second order used as basis functions.
- Calculations were made several times for various values of τ.

. . . Example . . .

Figure: Decay of maximal relative error



(a) Logarithm of maximal relative error in time of h for

(b) Logarithm of maximal relative error in time of u for

Inverse source problem . . .

[Šišková and Slodička, 2017] APNUM

Governing PDE ($x \in \Omega \subset \mathbb{R}^d,\ t \in (0,T)$)

$$\left(g_{2-\beta} * \partial_{tt} u(x)\right)(t) - \Delta u(x,t) = h(t)f(x) + F(x,t,u(x,t)) \tag{12}$$

BC & ICs

$$\begin{aligned} u(x,0) &= u_0(x), & x \in \Omega, \\ \partial_t u(x,0) &= v_0(x), & x \in \Omega, \\ -\nabla u(x,t)\cdot \boldsymbol{\nu} &= g(x,t), & (x,t) \in \Gamma \times (0,T), \end{aligned} \tag{13}$$

The unknown time-dependent function $h(t)$ will be determined from the following additional measurement

$$\int_\Omega u(x,t)\omega(x)\,\mathrm{d}x = m(t), \qquad t \in [0,T], \tag{14}$$

Variational setting

Multiplying (12) by the function ω, integrating over Ω, applying the Green theorem and using (14), we obtain

$$(g_{2-\beta} * m'')(t) + (\nabla u(t), \nabla \omega) = h(t)(f, \omega) - (g(t), \omega)_\Gamma + (F(t, u(t)), \omega) . \qquad \text{(MP)}$$

Similarly multiplying (12) by a function $\varphi \in H^1(\Omega)$ and using Green's theorem, we obtain the variational formulation of (12) and (13)

$$((g_{2-\beta} * \partial_{tt} u)(t), \varphi) + (\nabla u(t), \nabla \varphi) = h(t)(f, \varphi) + (F(t, u(t)), \varphi) - (g(t), \varphi)_\Gamma , \qquad \text{(P)}$$

for any $\varphi \in H^1(\Omega)$, a.a. $t \in [0, T]$ and $u(0) = u_0$, $\partial_t u(0) = v_0$.
The relations (P) and (MP) represent the variational formulation of the ISP (12), (13) and (14).

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Time discretization

The discrete convolution is defined by

$$(K * v)_i := \sum_{k=1}^{i} K_{i+1-k} v_k \tau,$$

note that by this definition we avoided problems with a blow up if K has a singularity at $t = 0$. Then we can calculate a difference for the discrete convolution as follows

$$\delta (K * v)_i = \frac{(K * v)_i - (K * v)_{i-1}}{\tau} = K_1 v_i + \sum_{k=1}^{i-1} \delta K_{i+1-k} v_k \tau, \qquad i \geq 1, \tag{15}$$

as

$$(K * v)_0 := 0$$

Schema

On the $i-$th time-layer we approximate the solution of (P),(MP) by (u_i, h_i), which solves

$$\left((g_{2-\beta} * \delta^2 u)_i, \varphi\right) + (\nabla u_i, \nabla \varphi) = h_i\,(f, \varphi) + (F(t_i, u_{i-1}), \varphi) - (g_i, \varphi)_\Gamma\,, \qquad \text{(DP}i\text{)}$$

for $\varphi \in H^1(\Omega)$, with $\delta u_0 := v_0$ and

$$(g_{2-\beta} * m'')_i + (\nabla u_{i-1}, \nabla \omega) = h_i\,(f, \omega) + (F(t_i, u_{i-1}), \omega) - (g_i, \omega)_\Gamma\,. \qquad \text{(DMP}i\text{)}$$

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Solvability of the ISP

Theorem 10

Let $f \in L^2(\Omega)$, $u_0, v_0, \omega \in H^1(\Omega)$, $\int_\Omega f\omega \neq 0$, $m \in C^3([0,T])$, and $g \in C^2([0,T],\Gamma)$. Suppose that F is a global Lipschitz continuous function in all variables and (11) holds true.
Then there exists a solution (u,h) to the (P), (MP) obeying $u \in C\left([0,T], H^1(\Omega)\right)$ with $\partial_t u \in C\left([0,T], L^2(\Omega)\right) \cap L^\infty\left((0,T), H^1(\Omega)\right)$, $\partial_{tt} u \in L^2\left((0,T), L^2(\Omega)\right)$ and $h \in C([0,T])$.

Theorem 11

Let $f, v_0 \in L^2(\Omega)$, $u_0, \omega \in H^1(\Omega)$, $\int_\Omega f\omega \neq 0$, $m \in C^2([0,T])$, F be a global Lipschitz continuous function in all variables and $g \in C([0,T],\Gamma)$. Then there exists at most one solution (u,h) to the (P), (MP) obeying $u \in C\left([0,T], H^1(\Omega)\right)$, $\partial_t u \in C\left([0,T], L^2(\Omega)\right) \cap L^2\left((0,T), H^1(\Omega)\right)$ with $\partial_{tt} u \in L^2\left((0,T), L^2(\Omega)\right)$ and $h \in C([0,T])$.

Fractional Dynamical BC

[Šišková and Slodička, 2018] CAMWA

$$(g_{2-\beta} * \partial_{tt} u(x))\,(t) - \Delta u(x,t) = h(t)f(x), \qquad x \in \Omega,\ t \in (0,T), \tag{16}$$

The equation (16) is accompanied with the following initial and boundary conditions

$$\begin{aligned} u(x,0) &= u_0(x), && x \in \Omega, \\ \partial_t u(x,0) &= v_0(x), && x \in \Omega, \\ u(x,t) &= 0, && (x,t) \in \Gamma_D \times (0,T), \\ -(g_{2-\beta} * \partial_{tt} u(x))\,(t) - \nabla u(x,t) \cdot \boldsymbol{\nu} &= \sigma(x,t), && (x,t) \in \Gamma_N \times (0,T), \end{aligned} \tag{17}$$

ISP: Find the couple (u, h) obeying

$$\int_\Omega u(x,t)\omega(x)\,\mathrm{d}x = m(t), \qquad t \in [0,T], \tag{18}$$

Evolutionary boundary condition

The dynamical BCs model non-perfect contact. (They are not very common in the mathematical literature.)

- They appear in many mathematical models including heat transfer in a solid in contact with a moving fluid, thermo-elasticity, diffusion phenomena, problems in fluid dynamics, etc. (see [Escher, 1993, Igbida and Kirane, 2002, Chill et al., 2006] and the references therein).
- $\boldsymbol{\nu} \times \boldsymbol{E} = \boldsymbol{\nu} \times (\partial_t \boldsymbol{B}(\boldsymbol{H}) \times \boldsymbol{\nu})$ [Vrábeľ and Slodička, 2012]
- $\partial_t \beta(u) + (\nabla u + \boldsymbol{b}(u)) \cdot \boldsymbol{\nu} = g$ [Su, 1993]
- $\partial_t \beta(u) - \Delta_\Gamma u + u + \nabla u \cdot \boldsymbol{\nu} = g$ [Vrábeľ and Slodička, 2013]

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Boundary measurement

[Šišková and Slodička, 2019] JCAM

$$\begin{cases} \left(g_{2-\beta} * \partial_{tt} u(x)\right)(t) - \Delta u(x,t) & = & h(t)f(x) + F(x,t,u(x,t)), \\ u(x,0) & = & u_0(x), \\ \partial_t u(x,0) & = & v_0(x), \\ -\nabla u(x,t) \cdot \boldsymbol{\nu} & = & \gamma(x,t) \end{cases} \tag{19}$$

The *Inverse Source Problem* (ISP) we are interested in here consists of identifying a couple $(u(x,t), h(t))$ obeying (19) and

$$\int_\Gamma u(x,t)\omega(x) dS = m(t), \qquad t \in [0,T], \tag{20}$$

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