Nakhushev extremum principle for a class of integro-differential operators

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Introduction *Fermat's extremum theorem*

$$f(x) \in C^1(a,b) \qquad y \in (a,b)$$

$$f(y) \geq f(x) \qquad a < x < b \qquad \Rightarrow \qquad f'(y) = 0$$

$$\not\Rightarrow \qquad \left(D^\alpha f\right)(y) = 0 \qquad \alpha \in (0,1)$$

$$f(y) \geq f(x) \qquad a < x < y \qquad \Rightarrow \qquad f'(y) \geq 0$$

$$\Rightarrow \qquad \left(D^\alpha f\right)(y) \geq 0$$

Introduction *Extremum principle for the Riemann-Liouville derivative*

$$\left(D_{ax}^{\alpha} f\right)(x) = \frac{d}{dx}\int_a^x f(t)\frac{(x-t)^{-\alpha}}{\Gamma(1-\alpha)}\,dt \qquad \alpha \in (0,1)$$

Theorem [A. M. Nakhushev, *Differential Equations*, 1974, vol. 10]

Let $f(x) \in L(a,y)$, $f(x) \in H^{\lambda}[y-\delta, y]$ $(\lambda > \alpha)$,

and $f(y) \geq f(x)$ $\forall x \in (a,y)$.

Then $$\left(D_{ax}^{\alpha} f\right)(y) \geq f(y)\frac{(y-a)^{-\alpha}}{\Gamma(1-\alpha)}.$$

In addition, if $f(y) > 0$ then $\left(D_{ax}^{\alpha} f\right)(y) > 0$.

- Mixed type PDE
- Loaded integral and differential equations
- Degenerate PDE
- Problems with shift for PDE
- Fractional differential equations

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Main results *Integro-differential operators*

$$\left(\mathbf{D}f\right)(x) = \frac{d}{dx}\int_a^x k(x,t)f(t)\,dt,$$

$$k : S \to \mathbb{R} \qquad f : (a,y) \to \mathbb{R} \qquad (\mathbf{D}f) : (b,y) \to \mathbb{R}$$

$$S = \{(x,t) :\ b < x < y,\ a < t < x\} \qquad a,\ b,\ y \in \mathbb{R} \cup \{-\infty\} \qquad -\infty \le a \le b < y$$

$$f(t) \in C(a,y] \cap L(a,y) \qquad [k(x,t)]_{x=y} \in L(a,y)$$

$$k(x,t),\ k_x(x,t) \in C\left(\overline{S} \setminus \{x=t\}\right)$$

Main results *Extremum principle*

Theorem

Let

$$[k_x(x,t)]_{x=y} \leq 0$$

and

$$\int_a^{x-\varepsilon} [f(t) - f(x)]\, k_x(x,t)\, dt \rightrightarrows h_0(x)$$

$$\varepsilon \to 0$$

$$\frac{d}{dx}\int_a^{x-\varepsilon} k(x,t)\, dt \rightrightarrows h_1(x) \qquad x \in (y-\delta, y]$$

$$[f(x-\varepsilon) - f(x)]\, k(x, x-\varepsilon) \rightrightarrows 0$$

Theorem *continuation*

Then

$$f(y) = \sup_{a<t<y} f(t) \qquad \Longrightarrow \qquad (\mathbf{D} f)(y) \geq f(y)\,(\mathbf{D}\,1)(y)$$

$$(\mathbf{D} f)(y) = f(y)\,(\mathbf{D}\,1)(y) \qquad \Longleftrightarrow \qquad \begin{array}{l} f(t) \equiv const \\ \text{or} \\ [k_x(x,t)]_{x=y} \equiv 0 \end{array}$$

and

$$\begin{array}{c} f(t) \not\equiv const \quad [k_x(x,t)]_{x=y} \not\equiv 0 \\ f(y)\,(\mathbf{D}\,1)(y) \geq 0 \end{array} \qquad \Longrightarrow \qquad (\mathbf{D}\, f)(y) > 0$$

Proof sketch

$$K_\varepsilon(x) = \int_a^{x-\varepsilon} k(x,t)\, f(t)\, dt \qquad (\varepsilon > 0).$$

$$K_\varepsilon(x) = \int_a^{x-\varepsilon} [f(t) - f(x)]\, k(x,t)\, dt + f(x) \int_a^{x-\varepsilon} k(x,t)\, dt$$

$$\frac{d}{dx} K_\varepsilon(x) = \int_a^{x-\varepsilon} [f(t) - f(x)]\, k_x(x,t)\, dt + f(x) \frac{d}{dx} \int_a^{x-\varepsilon} k(x,t)\, dt$$

$$(\mathbf{D}\, f)\,(x) = \lim_{\varepsilon \to 0} \frac{d}{dx} K_\varepsilon(x) = \int_a^x [f(t) - f(x)]\; k_x(x,t)\, dt + f(x)\, (\mathbf{D}\, \mathsf{1})\, (x).$$

Remark

$$\mathcal{D}^{RL} = D\,I \qquad \text{Riemann-Liouville type}$$

$$\mathcal{D}^{C} = I\,D \qquad \text{Caputo type}$$

$$\mathcal{D}^{C} u + Lu = F \qquad u \in C(\overline{\Omega})$$

$$\mathcal{D}^{RL} u + Lu = F \qquad u \notin C(\overline{\Omega}) \qquad \varphi \cdot u \in C(\overline{\Omega})$$

Weighted extremum principle

Let $\varphi(t) \in C(a,y] \cap L(a,y)$ and

$$f(t) = \varphi(t)\, g(t) \qquad g(t) \in C[a,y]$$

$$\varphi(t)\, [k_x(x,t)]_{x=y} \leq 0$$

and

$$\int_a^{x-\varepsilon} \varphi(t)\, [g(t) - g(x)]\, k_x(x,t)\, dt \rightrightarrows h_0(x)$$

$$\varepsilon \to 0$$

$$\frac{d}{dx} \int_a^{x-\varepsilon} \varphi(t)\, k(x,t)\, dt \rightrightarrows h_1(x) \qquad x \in (y-\delta, y]$$

$$[g(x-\varepsilon) - g(x)]\, k(x, x-\varepsilon) \rightrightarrows 0$$

Weighted extremum principle *continuation*

Then

$$g(y) = \sup_{a<t<y} g(t) \quad \Longrightarrow \quad (\mathbf{D}f)(y) \geq g(y)\,(\mathbf{D}\,\varphi)(y)$$

$$(\mathbf{D}f)(y) = g(y)\,(\mathbf{D}\,\varphi)(y) \quad \Longleftrightarrow \quad \begin{array}{l} g(t) \equiv const \\ \text{or} \\ \varphi(t)\,[k_x(x,t)]_{x=y} \equiv 0 \end{array}$$

and

$$\begin{array}{c} g(t) \not\equiv const \quad \varphi(t)\,[k_x(x,t)]_{x=y} \not\equiv 0 \\ g(y)\,(\mathbf{D}\,\varphi)(y) \geq 0 \end{array} \quad \Longrightarrow \quad (\mathbf{D}\,f)(y) > 0$$

Proof sketch

By

$$f(t) = \varphi(t)\, g(t)$$

we get

$$(\mathbf{D}\, f)\,(x) = \frac{d}{dx} \int_a^x f(t)\, k(x,t)\, dt = \frac{d}{dx} \int_a^x g(t)\, \tilde{k}(x,t)\, dt = \big(\tilde{\mathbf{D}}\, g\big)(x),$$

where

$$\tilde{k}(x,t) = \varphi(t) k(x,t).$$

Weighted extremum principle for the Riemann-Liouville fractional derivative

Let $\alpha \in (0,1)$ and

$$t^{1-\alpha} f(t) \in C[0,y] \qquad f(t) \in H^{\lambda}[y-\delta, y] \qquad (\lambda > \alpha, \quad \delta > 0)$$

and

$$t^{1-\alpha} f(t) \le y^{1-\alpha} f(y) \qquad \forall t \in (0,y)$$

Then

$$\left(D^{\alpha}_{0x} f\right)(y) \ge 0$$

and

$$\left(D^{\alpha}_{0x} f\right)(y) = 0 \qquad \Longleftrightarrow \qquad f(t) = const \cdot t^{\alpha-1}$$

Proof sketch

Taking

$$k(x,t) = \frac{(x-t)^{-\alpha}}{\Gamma(1-\alpha)} \qquad \text{and} \qquad \varphi(t) = t^{\alpha-1}$$

we get

$$g(t) = t^{1-\alpha} f(t) \qquad (\mathbf{D}\, f)\,(x) = \left(D_{0x}^{\alpha} f\right)(x) \qquad \text{and} \qquad \left(D_{0x}^{\alpha}\, \varphi\right)(x) = 0$$

Therefore

$$\left(D_{0x}^{\alpha} f\right)(y) = \left(D_{0x}^{\alpha}\, \varphi g\right)(y) \geq g(y) \left(D_{0x}^{\alpha}\, \varphi\right)(y) = 0$$

Application example *Problem statement*

$$\left(\frac{\partial^\alpha}{\partial y^\alpha} - \frac{\partial^2}{\partial x^2}\right) u(x,y) = f(x,y) \qquad (0 < \alpha < 1)$$

$$\Omega = \{(x,y): \quad z_1(y) < x < z_2(y), \quad 0 < y < T\}$$

$$u(z_1(y),y) = \varphi_1(y) \qquad u(z_2(y),y) = \varphi_2(y) \qquad 0 < y < T$$

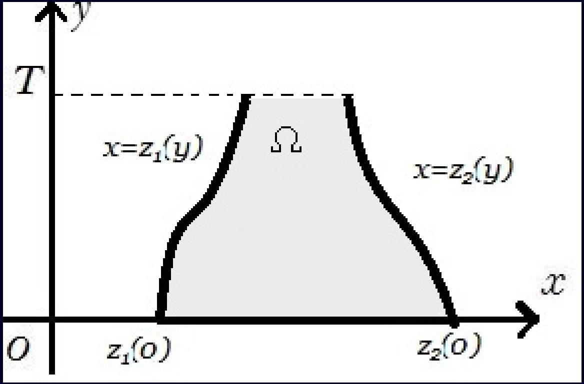
$$\lim_{y \to 0} y^{1-\alpha} u(x,y) = \tau(x) \qquad z_1(0) < x < z_2(0)$$

$$y^{1-\alpha} u(x,y) \in C(\overline{\Omega})$$

Application example *Domain*

$$\Omega = \{(x,y) :\ z_1(y) < x < z_2(y),\, 0 < y < T\}$$

$$z_1(y) \nearrow \qquad z_2(y) \searrow \qquad z_1(y) < z_2(y)$$



Application example *Uniqueness of solutions*

Let $u(x,y)$ be a solution of homogeneous problem ($f \equiv 0, \quad \tau \equiv 0, \quad \varphi_i \equiv 0$) and

$$v(x,y) = y^{1-\alpha} u(x,y) \qquad\qquad v(x,y) \in C(\overline{\Omega})$$

$$v(x,y) \not\equiv 0 \quad \Longrightarrow \quad \exists\,(\xi,\eta) \in \overline{\Omega}: \quad v(\xi,\eta) = \sup_{\Omega} v(x,y) > 0 \quad (\text{otherwise} \quad v \to -v)$$

$$(\xi,\eta) \notin \partial\Omega \backslash \{y=T\} \quad \Longrightarrow \quad \begin{matrix} \left[\frac{\partial^2}{\partial x^2} v\right]_{(\xi,\eta)} \le 0 \quad \left[\frac{\partial^2}{\partial x^2} u\right]_{(\xi,\eta)} \le 0 \\ \\ \left[\frac{\partial^\alpha}{\partial y^\alpha} u\right]_{(\xi,\eta)} > 0 \quad \text{or} \quad u(x,y) = A \cdot y^{\alpha-1} \end{matrix} \quad \Longrightarrow \quad u \equiv 0$$

Convolution operators

Consider $\left(\mathbf{D}f\right)(x)$ with $k(x,t) = k(x-t)$ and $a = 0$ i.e.

$$\left(\mathbf{K}f\right)(x) = \frac{d}{dx}\int_a^x k(x-t)f(t)\,dt,$$

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Extremum principle for convolution operators

Let

$$f(t) \in C(0,y] \cap L(0,y) \qquad k(t) \in L(0,y) \cap C^1(0,y] \qquad k'(t) \le 0$$

$$\omega(t)\,k'(t) \in L(0,y) \qquad \lim_{\varepsilon\to 0} \omega(\varepsilon)\,k(\varepsilon) = 0 \qquad \omega(t) = \sup_{t<x<y} |f(x) - f(x-t)|$$

Then

$$f(y) \ge f(t) \quad \forall t \in (0,y) \qquad \Longrightarrow (\mathbf{K}\,f)\,(y) \ge f(y)k(y)$$

and

$$(\mathbf{K}\,u)\,(y) = f(y)h(y) \qquad \Longleftrightarrow \qquad f(t) \equiv const \quad \text{or} \quad k(t) \equiv const$$

Proof $\quad (\mathbf{K}\,1)\,(x) = \frac{d}{x}\int_0^x k(x-t)\,dt = k(x)$

Fractional derivative of a function with respect to another function

Let $\alpha \in (0,1)$, $g(t) \in C^1[0,y]$, and $g'(t) > 0$ $\forall t \in [0,y]$

$$\left(\mathcal{D}_g^\alpha f\right)(x) = \frac{1}{g'(x)\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{f(t)g'(t)}{[g(x)-g(t)]^\alpha}dt$$

$f(t) \in H^\lambda[y-\delta, y]$ $\lambda > \alpha$

$f(y) \geq f(t)$ $\forall t \in [a,y]$

$$\Longrightarrow \quad \left(\mathcal{D}_g^\alpha f\right)(y) \geq f(y)\frac{[g(y)-g(a)]^{-\alpha}}{\Gamma(1-\alpha)}$$

Proof

$$\left(\mathcal{D}_g^\alpha f\right)(x) = \frac{1}{g'(x)}\left(\mathbf{D} f\right)(x) \quad \text{where} \quad k(x,t) = \frac{g'(t)}{\Gamma(1-\alpha)}\left[g(x)-g(t)\right]^{-\alpha}$$

Thank you!