Workshop: Fractional Calculus
International Conference on Fractional Calculus
(Ghent University, 9 – 10 June 2020)

"Ghent Analysis & PDEs" Res. Group

"THE MULTI-INDEX MITTAG-LEFFLER FUNCTIONS AS SPECIAL FUNCTIONS OF FRACTIONAL CALCULUS: EXAMPLES AND APPLICATIONS"

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The developments in theoretical and applied science require knowledge of the properties of Mathematical Functions (following the notion from Wolfram Res. site, https://functions.wolfram.com/), from elementary trigonometric functions to the multivariety of Special Functions (SF), appearing whenever natural and social phenomena are studied, engineering problems are formulated, and numerical simulations are performed. This survey aims to attract attention to classes of SF that were not so popular (or some of them not introduced) until Fractional Calculus (FC) gained the important role, related to the boom of applications of fractional order models.

The so-called Special Functions of Fractional Calculus (SF of FC) are basically Fox H-functions, among them are the generalized Wright hypergeometric functions ${}_p\Psi_q$ and in particular, the classical Mittag-Leffler (M-L) functions and their various extensions. The SF of FC are now unavoidable tool in solutions of fractional order integral and differential equations and systems.

A very general class of the so-called multi-index (vector index) M-L functions have been recently introduced (Luchko, Kiryakova) and studied in details. Some of their basic properties, as results of our works, are discussed.

An unexpectedly long list of SF of FC with various applications in fractional order models are shown to fall as particular cases of the multi-index M-L functions:

– The 1- and 2- parameter M-L functions and related special cases (as the error-, incomplete gamma, Rabotnov functions);
– Prabhakar (3-parameter) function;
– Wright (Bessel-Maitland) function;
– Dzhrbasjan's 2×2-indices functions;
– Mainardi function;
– Bessel, Struve, Lommel and Airy functions;
– Delerue / Kljuchantsev hyper-Bessel functions;
– several other M-L type functions with 3, 4, etc. indices
- by Pathak, Oteiza-Kalla-Conde, Kilbas-Srivastava-Trujillo, Paneva-Konovska, etc.

In this lecture we briefly survey some of our basic results on SF of FC, so to attract attention of potential users:
– all of them can be represented as GFC operators of basic elementary functions (among them the pdf functions of some probability distributions like gamma-, beta-, etc.);
– classification and basic properties;
– evaluation of FC and GFC operators of these SF in the general case;
– numerous examples. For their extended list and details, see Refs.

Here, under the term operators of Generalized Fractional Calculus (GFC) we mean integral, differ-integral and integro-differential operators which involve SF with singularities in the kernels, have convolutional structure and satisfy the basic "axioms" of the classical FC. The notion "generalized operators of fractional integration" was introduced by Kalla, 1969-1979. He suggested their general form, studied some formal properties and examples when the kernels were SF as the Gauss and generalized hypergeometric functions, including arbitrary G- and H-functions.

The ideas of S.L. Kalla provoked the author to choose a more peculiar case of G- and H-kernels ($G^{m,n}_{m+n,m+n}$ or $H^{m,n}_{m+n,m+n}$) and this choice allowed developing a full theory of the corresponding GFC with many particular cases and applications:

• V. Kiryakova, *Generalized Fractional Calculus and Applications*. Longman & J. Wiley, 1994.

As particular cases of GFC operators, they appear:

the operators of classical FC ($m = 1$) of Riemann-Liouville and Caputo, Erdélyi-Kober (E-K) operators and their various compositions, named as:
Saigo ($m = 2$), Marichev-Saigo-Maeda ($m = 3$), hyper-Bessel operators (Dimovski) ($m \geq 1$, arbitrary integer), etc.,

as well as many other operators of generalized integration and differentiation studied and used in Calculus, Differential and Integral Equations and their applications in mathematical models.

The *Wright generalized hypergeometric function* ${}_p\Psi_q(z)$, called also *Fox-Wright function* (F-W g.h.f.) is defined as:

$$ {}_p\Psi_q \left[\begin{array}{c} (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q) \end{array} \middle| z \right] = \sum_{k=0}^{\infty} \frac{\Gamma(a_1 + kA_1) \ldots \Gamma(a_p + kA_p)}{\Gamma(b_1 + kB_1) \ldots \Gamma(b_q + kB_q)} \frac{z^k}{k!} \tag{1} $$

$$ = H^{1,p}_{p,q+1} \left[-z \middle| \begin{array}{c} (1 - a_1, A_1), \ldots, (1 - a_p, A_p) \\ (0, 1), (1 - b_1, B_1), \ldots, (1 - b_q, B_q) \end{array} \right], \tag{2} $$

in terms of the more general Fox H-function. Denote

$$ \rho = \prod_{i=1}^{p} A_i^{-A_i} \prod_{j=1}^{q} B_j^{B_j}, \quad \Delta = \sum_{k=1}^{j} B_j - \sum_{i=1}^{p} A_i. $$

If $\Delta > -1$, the ${}_p\Psi_q$-function is an entire function of z, $z \in \mathbb{C}$, and if $\Delta = -1$, this series is absolutely convergent in the disk $\{|z| < \rho\}$, and for $|z| = \rho$ if

$$ \Re(\mu) = \Re \left\{ \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p-q}{2} \right\} > 1/2. $$

For $A_1 = \cdots = A_p = 1, B_1 = \cdots = B_q = 1$ in (1) and (2), the Wright g.h.f. reduces to a *generalized hypergeometric* ${}_pF_q$*-function*, and so also to a Meijer's G-function

$$
\begin{aligned}
{}_p\Psi_q \left[\begin{array}{c} (a_1, 1), \ldots, (a_p, 1) \\ (b_1, 1), \ldots, (b_q, 1) \end{array} \middle| z \right] &= c\, {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q;\ z) \\
&= c \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{z^k}{k!} \qquad (3) \\
&= c\, G^{1,p}_{p,q+1} \left[-z \middle| \begin{array}{c} 1-a_1, \ldots, 1-a_p \\ 0, 1-b_1, \ldots, 1-b_q \end{array} \right];
\end{aligned}
$$

$$
\text{with} \quad c = \left[\prod_{i=1}^{p} \Gamma(a_i) \Big/ \prod_{j=1}^{q} \Gamma(b_j) \right], \ (a)_k := \Gamma(a+k)/\Gamma(a).
$$

Operators of Generalized Fractional Calculus

Generalized Fractional Calculus (GFC): first, *generalized fractional integration operators* are considered by Kalla as a common notion, in the form:

$$\mathcal{I}\, f(z) = \int_0^1 \Phi(\sigma)\, \sigma^{\gamma}\, f(z\sigma) d\sigma = z^{-\gamma-1} \int_0^z \Phi(\frac{\xi}{z}) \xi^{\gamma} f(\xi) d\xi, \qquad (4)$$

where $\Phi(\sigma)$ is suitably chosen continuous/ analytical function, so that above integral makes sense and satisfies the main axioms of the classical FC. To have a good theory of GFC, the adequate choice of such kernel-function is indeed crucial : NOT to be too general, neither to be too particular, and so that one can determine *what is the fractional order of integration*.

For too general kernel SF, NOT too much formal properties were derived, as a formal (only) inversion formula (by means of the Mellin transform) and NO full theory and applications were developed. So, it was necessary to make a suitable choice of the *G*- and *H*-kernel function, so to develop a full theory of GFC with determined fractional order (multi-order) and its applications.

Kiryakova's generalized fractional integrals

The *multiple E-K integral* (of multiplicity $m > 1$), is defined by means of the real parameters' sets: $(\delta_1, ..., \delta_m)$ – *multi-order of integration*, $(\gamma_1, ..., \gamma_m)$ – multi-weight; and $(\beta_1 > 0, ..., \beta_m > 0)$ – additional multi-parameter, as:

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(z) := \int_0^1 H_{m,m}^{m,0} \left[\sigma \,\middle|\, \begin{matrix} (\gamma_k + \delta_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \\ (\gamma_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \end{matrix} \right] f(z\sigma) d\sigma, \tag{5}$$

if $\sum_{k=1}^{m} \delta_k > 0$; and as the identity operator:

$I_{(\beta_k),m}^{(\gamma_k),(0,...,0)} f(z) = f(z)$, if $\delta_1 = \delta_2 = \cdots = \delta_m = 0$. Note that the kernel $H_{m,m}^{m,0}$-function (with 3 singularities: 0, 1, ∞) is analytic function in the unit disk and $H_{m,m}^{m,0}(\sigma) \equiv 0$ for $|\sigma| > 1$.

If all β's are equal: $\beta_1 = \beta_2 = ... = \beta_m = \beta > 0$, (5) reduces to a simpler representation by means of *Meijer's G-function*,

$$I_{(\beta,...,\beta),m}^{(\gamma_k),(\delta_k)} f(z) := I_{\beta,m}^{(\gamma_k),(\delta_k)} f(z) = \int_0^1 G_{m,m}^{m,0} \left[\sigma \,\middle|\, \begin{matrix} (\gamma_k + \delta_k)_1^m \\ (\gamma_k)_1^m \end{matrix} \right] f(z\sigma^{1/\beta}) d\sigma. \tag{6}$$

We call all the operators of the form

$$\widetilde{I}f(z) = z^{\delta_0}\, I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(z),\ \ \widetilde{I}f(z) = z^{\delta_0}\, I^{(\gamma_k),(\beta_k)}_{\beta,m} f(z),\ \text{ with some } \delta_0 \geq 0, \tag{7}$$

as *generalized fractional integrals* of *multi-order* $(\delta_1, ..., \delta_m)$.

The corresponding generalized fractional derivatives (of R-L type) are defined by means of the differ-integral operator:

$$D^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(z) := D_\eta\, I^{(\gamma_k+\delta_k),(\eta_k-\delta_k)}_{(\beta_k),m} f(z) \tag{8}$$

$$= D_\eta \int_0^1 H^{m,0}_{m,m}\left[\sigma \left| \begin{array}{c} (\gamma_k + \eta_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \\ (\gamma_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \end{array} \right.\right] f(z\sigma)\, d\sigma,$$

where the auxiliary (integer order) differential operator D_η is defined as

$$D_\eta = \left[\prod_{r=1}^{m} \prod_{j=1}^{\eta_r} \left(\frac{1}{\beta_r} z \frac{d}{dz} + \gamma_r + j\right)\right], \tag{9}$$

by means of integers $\eta_k = [\delta_k] + 1$, or $\eta_k = \delta_k$ if integer, $k = 1, ..., m$.

And the Caputo-type g.f.d. is: ${}^*D^{(\gamma_k),(\delta_k)}_{(\beta_k),m} = I^{(\gamma_k+\delta_k),(\eta_k-\delta_k)}_{(\beta_k),m} D_\eta$.

For $\alpha \geq \max_k[-\beta_k(\gamma_k+1)]$, say in C_α, up to a constant multiplier,

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left\{z^p\right\} = c_p z^p, \; p > \alpha, \;\; \text{where} \;\; c_p = \prod_{k=1}^{m} \frac{\Gamma(\gamma_k + \frac{p}{\beta_k} + 1)}{\Gamma(\gamma_k + \delta_k + \frac{p}{\beta_k} + 1)},$$

and it is an invertible mapping $I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} : C_\alpha \mapsto C_\alpha^{(\eta_1+\cdots+\eta_m)} \subset C_\alpha$.
Analogously, under the same conditions, (5) maps the class of weighted analytic functions $H_\alpha(\Omega)$ into itself, preserving the power functions (up to constant multipliers like above) and the image of a power series has the same radius of convergence.

It is also shown that (5) has a Mellin type convolutional representation, based on its Mellin image.

Another well expected result, for Lebesgue integrable functions is the following: (5) is a bounded linear operator from the Banach space $L_{\alpha,p}(0,\infty)$ into itself:

$$\left\| I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f \right\|_{\alpha,p} \leq h_{\alpha,p}\, \|f\|_{\alpha,p}, \quad \text{i.e.} \quad \left\| I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f \right\| \leq h_{\alpha,p}$$

with $h_{\alpha,p} = \ldots$.

The following operational rules confirm that *the operators of our GFC satisfy the axioms of FC*:

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\{\lambda f(cz)+\eta g(cz)\}=\lambda\left\{I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}f\right\}(cz)+\eta\left\{I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}g\right\}(cz)$$

(bilinearity of (5));

$$I_{(\beta_1,\ldots,\beta_m),m}^{(\gamma_1,\ldots,\gamma_s,\gamma_{s+1},\ldots,\gamma_m),(0,\ldots,0,\delta_{s+1},\ldots,\delta_m)}f(z)=I_{(\beta_{s+1},\ldots,\beta_m),m-s}^{(\gamma_{s+1},\ldots,\gamma_m)(\delta_{s+1},\ldots,\delta_m)}f(z)$$

(i.e., if $\delta_1=\delta_2=\ldots=\delta_s=0$, then the multiplicity reduces to $(m-s)$);

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}z^{\lambda}f(z)=z^{\lambda}I_{(\beta_k),m}^{(\gamma_k+\frac{\lambda}{\beta_k}),(\delta_k)}f(z),\quad \lambda\in\mathbb{R}$$

(generalized commutability with power functions);

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}I_{(\varepsilon_j),n}^{(\tau_j),(\alpha_j)}f(z)=I_{(\varepsilon_j),n}^{(\tau_j),(\alpha_j)}I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}f(z)$$

(commutability of operators of form (5));

$$\text{the left-hand side of above}=I_{((\beta_k)_1^m,(\varepsilon_j)_1^n),m+n}^{((\gamma_k)_1^m,(\tau_j)_1^n)((\delta_k)_1^m,(\alpha_j)_1^n)}f(z)$$

(compositions of m-tuple and n-tuple integrals (5) give $(m+n)$-tuple integrals of same form);

$$I_{(\beta_k),m}^{(\gamma_k+\delta_k),(\sigma_k)} I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(z) = I_{(\beta_k),m}^{(\gamma_k),(\sigma_k+\delta_k)} f(z), \quad \delta_k > 0, \sigma_k > 0,$$

(law of indices, product rule or semigroup property);

$$\left\{ I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} \right\}^{-1} f(z) = I_{(\beta_k),m}^{(\gamma_k+\delta_k),(-\delta_k)} f(z)$$

(formal inversion formula).

This inversion follows from the above index law for $\sigma_k = -\delta_k < 0$, $k = 1, ..., m$ and definition for zero multi-order of integration:

$$I_{(\beta_k),m}^{(\gamma_k+\delta_k),(-\delta_k)} I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(z) = I_{\beta_k,m}^{(\gamma_k),(0,...,0)} f(z) = f(z).$$

But the symbols (5) are not defined for negative multi-orders of integration $-\delta_k < 0$, $k = 1, ..., m$. The problem was to propose an appropriate meaning for them and so to avoid the appearance of divergent integrals, similarly to the classical case when the R-L and E-K integrals have to be inverted by additional differentiation of suitable integer order $\eta = [\delta] + 1$, thus defining the R-L or Caputo FDs. In GFC, the problem was more difficult and resolved by auxiliary differential operator D_η, polynomial of $z\dfrac{d}{dz}$, using a set of integers $\eta = (\eta_1, ..., \eta_m)$ related to $\delta = (\delta_1, ..., \delta_m)$.

Some special cases of GFC operators

$\underline{m=1}$: Erdélyi-Kober operators. E-K integrals:

$$I_\beta^{\gamma,\delta} f(z) = \frac{1}{\Gamma(\delta)} \int\limits_0^1 \sigma^\gamma (1-\sigma)^{\delta-1}\ f(z\sigma^{\frac{1}{\beta}})\ d\sigma, \qquad (10)$$

and E-K type integrals: $\widetilde{I} f(z) = z^{\delta_0} I_\beta^{\gamma,\delta} f(z),\ \ \delta_0 \geq 0.$

E-K derivatives: $D_\beta^{\gamma,\delta} f(z) = D_n\, I_\beta^{\gamma+\delta, n-\delta} f(z)$ (11)

$$= \prod_{j=1}^{n} \left(\frac{1}{\beta} z \frac{d}{dz} + \gamma + j \right) I_\beta^{\gamma+\delta, n-\delta} f(z), \quad n-1 < \delta \leq n,\ n \in \mathbb{N}.$$

If $\gamma = 0$, $\beta = 1$, the E-K operators reduce to Riemann-Liouville (R-L) operators:

$$R^\delta f(z) = z^\delta\, I_1^{0,\delta} f(z), \quad D^\delta f(z) = z^{-\delta} D_1^{-\delta,\delta} f(z); \qquad (12)$$

conversely, $I_1^{\gamma,\delta} f(z) = z^{-\gamma-\delta} R^\delta z^\gamma f(z)$.

Also, the corresponding Caputo-type derivatives come as special cases.

Important property: Composition/ decomposition:

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(z) = \left[\prod_{k=1}^{m} I^{\gamma_k,\delta_k}_{\beta_k} \right] f(z) \tag{13}$$

$$= \int_0^1 \cdots \int_0^1 \left[\prod_{k=1}^{m} \frac{(1-\sigma_k)^{\delta_k - 1} \sigma_k^{\gamma_k}}{\Gamma(\delta_k)} \right] f\left(z\, \sigma_1^{1/\beta_1} \ldots \sigma_m^{1/\beta_m} \right) d\sigma_1 \ldots d\sigma_m,$$

and similarly (sequential derivatives),

$$D^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(z) = D^{\gamma_1,\delta_1}_{\beta_1} \left\{ \ldots D^{\gamma_m,\delta_m}_{\beta_m} \right\} f(z).$$

$\underline{m=2}$: Hypergeometric fractional integrals with kernel $H^{2,0}_{2,2} = {}_2F_1$, the Gauss function. We have several examples, like the most general Hohlov operators $\mathcal{F} f(z) = {}_2F_1(a, b; c; z) \circ f(z)$ (in univalent function theory), and in particular the Saigo operators with "our" $\beta_1 = \beta_2 = 1$:

$$I^{\alpha,\beta,\eta} f(z) = \frac{z^{-\beta}}{\Gamma(\alpha)} \int_0^1 (1-\sigma)^{\alpha-1} \, {}_2F_1(\alpha+\beta, -\eta; \alpha; 1-\sigma) \, f(z\sigma) d\sigma. \tag{14}$$

But these GFC operators with $\underline{m=2}$ are also compositions of two E-K operators, or of two weighted R-L integrals:

$$I^{\alpha,\beta,\eta}f(z) = z^{-\beta}\, I_{(1,1),2}^{(\eta-\beta,0),(-\eta,\alpha+\eta)} f(z) = z^{-\beta}\, I_1^{\eta-\beta,-\eta}\, I_1^{0,\alpha+\eta}\, f(z)$$

$$= I_1^{\eta,-\eta}\, I_1^{\beta,\alpha+\eta}\, z^{-\beta} f(z) = I_{(1,1),2}^{(\eta,\beta),(-\eta,\alpha+\eta)}\, z^{-\beta} f(z) \qquad (15)$$

$$= R^{-\eta}\, z^{-\alpha-\beta}\, R^{\alpha+\eta} f(z).$$

The corresponding Saigo fractional derivative has explicit differ-integral expression (Kiryakova, 1994) as

$$D^{\alpha,\beta,\eta} f(z) = (d/dz)^n\, I^{\alpha+n,\beta-n,\eta-n} f(z) \ \text{ with } \ n = [-\Re\alpha]+1,$$

It is also a GFC operator and composition of 2 E-K derivatives:

$$D^{\alpha,\beta,\eta} f(z) = z^{\beta} D_{(1,1),2}^{(\eta,\beta),(-\eta,\alpha+\eta)} f(z) = z^{\beta} D_1^{\eta,-\eta} D_1^{\beta,\alpha+\eta} f(z).$$

$\underline{m=3}$: Marichev (1974)-Saigo-Maeda (1985, 1996) operators (M-S-M)

Let a, a', b, b', c be complex parameters, $\Re c > 0$,

$$I^{a,a',b,b',c}f(z) = \frac{z^{-a}}{\Gamma(c)} \int_0^z (z-\xi)^{c-1} \xi^{-a'} F_3(a, a', b, b'; c; 1-\frac{\xi}{z}, 1-\frac{z}{\xi}) f(\xi) d\xi$$

$$= z^{c-a-a'} \int_0^1 \frac{(1-\sigma)^{c-1}}{\Gamma(c)} \sigma^{-a'} F_3(a, a', b, b'; c; 1-\sigma, 1-\frac{1}{\sigma}) f(z\sigma) d\sigma. \tag{16}$$

The kernel-function, the so-called *Appel F_3 function* (*Horn function*), $|z| < 1$, $|\xi| < 1$:

$$F_3\left(a, a', b, b', c, z, \xi\right) = \sum_{m,n=0}^{\infty} \frac{(a)_m (a')_n (b)_m (b')_n}{(c)_{m+n}} \frac{z^m \xi^n}{m! n!},$$

is however a $H^{3,0}_{3,3}$, and even the simpler $G^{3,0}_{3,3}$, the kernel of GFC integral with $m = 3$:

$$G^{3,0}_{3,3}\left[\sigma \left| \begin{matrix} \cdots \\ \cdots \end{matrix} \right.\right] = \frac{(1-\sigma)^{c-1}}{\Gamma(c)} F_3\left(a, a', b, b', c, 1-\frac{1}{\sigma}, 1-\sigma\right).$$

Therefore, the M-S-M integrals are GFC integrals with $m = 3$, that is, *compositions of 3 commutable classical Erdélyi-Kober integrals* (see Kiryakova, 1994), a fact seemingly *unknown* to many other authors (or just ignored):

$$I^{a,a',b,b',c} f(z) = z^{c-a-a'} I_{(1,1,1),3}^{(a-a',b-a',c-2a'-b'),(b,c-a'-b,a')} f(z)$$

$$= z^{c-a-a'}\, I_1^{a-a',b}\, I_1^{b-a',c-a'-b}\, I_1^{c-2a'-b',a'}\, f(z). \tag{17}$$

Saigo and Maeda, and authors after them, considered also generalized M-S-M fractional derivatives $D^{a,a',b,b',c}$, and originally defined analogously to the Saigo derivatives $D^{\alpha,\beta,\eta}$. But these are special cases of 3-tuple generalized fractional derivatives in our sense (8), namely:

$$D^{a,a',b,b',c} f(z) = D_{(1,1,1),3}^{(a-a',b-a',c-2a'-b'),(b,c-a'-b,a')}\, z^{-c} f(z).$$

Then, all properties the other authors derive follow at once from our GFC theory!

$m \geq 2$, arbitrary: the hyper-Bessel operators (Dimovski (1966)) the Gel'fond-Leont'ev operators (w.r.t. multi-index M-L function, Kiryakova (1996)), etc.

The hyper-Bessel differential operators are interesting example of generalized "fractional" derivatives but of integer multi-order $(\delta_1, \delta_2, ..., \delta_m) = (1, 1, ..., 1)$. These are singular differential operators of higher integer order $m \geq 1$ with variable coefficients

$$B = z^{\alpha_0} \frac{d}{dz} z^{\alpha_1} \frac{d}{dz} \cdots z^{\alpha_{m-1}} \frac{d}{dz} z^{\alpha_m} , \quad 0 < z < \infty , \qquad (18)$$

and some real parameters $\alpha_k, k = 1, ..., m$ and $\beta = m - (\alpha_0 + \alpha_1 + \cdots + \alpha_m) > 0$. They have also alternative representations in the following equivalent forms, either as:

$$B = z^{-\beta} \prod_{k=1}^{m} \left(z \frac{d}{dz} + \beta\gamma_k \right) = z^{-\beta} Q_m(z \frac{d}{dz}), \quad 0 < z < \infty, \quad (19)$$

symmetric with respect to the zeros $\mu_k = -\beta\gamma_k,\ k = 1, \ldots, m$ of the m-th degree polynomial $Q_m(\mu)$, where $\gamma_k = \frac{1}{\beta} (\alpha_k + \cdots + \alpha_m - m + k),\ \ k = 1, ..., m;$

or as

$$B = z^{-\beta}\left[z^m \frac{d^m}{dz^m} + a_1 z^{m-1}\frac{d^{m-1}}{dz^{m-1}} + \cdots + a_m\right].$$

The latter gives better impression on the nature of the h.-B. differential operators, appearing often in problems of mathematical physics, and extending the Bessel differential operator B_ν of 2nd order $(m=2)$ with $\gamma_{1,2} = \pm\nu/2$, $\beta = 2$.

However, (31) and its linear right inverse integral operator L (hyper-Bessel integral operator) are examples of GFC operators:

$$B = z^{-1} D^{(\gamma_k - 1),(1,1,\ldots,1)}_{(\beta,\beta,\ldots,\beta),m}, \quad L = z\, I^{(\gamma_k),(1,1,\ldots,1)}_{(\beta,\beta,\ldots,\beta),m}. \tag{20}$$

In 1968, Dimovski considered also *fractional powers of the integral operator* L, namely the operators L^λ, $\lambda > 0$. To express them, he used the *notion of convolution* (the basic one in his "Convolutional Calculus", 1990). He represented the fractional powers of L as convolutional products, where $(*)$ had a very complicated form:

$$L^\lambda f = \{l_\lambda\} * f, \;\text{ with }\; l_\lambda = \Big\{\frac{t^{\beta(\lambda-\delta-1)}}{\prod_{k=1}^{m}\Gamma(\lambda-\delta+\gamma_k)}\Big\}, \;\; \delta \geq \max_k \gamma_k. \tag{21}$$

And it was proved that under this definition, the FC semigroup property is satisfied: $L^\lambda L^\mu = L^{\lambda+\mu}$, $\lambda > 0, \mu > 0$, $L^n = L \cdot L \cdots L$ for $n \in \mathbb{N}$.

However, from our point of view of SF, based on the G-functions, we found (1983) a representation similar to its form in (20) (where $\lambda = 1$), for example

$$L^\lambda f(z) = \frac{z^{\beta\lambda}}{(\beta\lambda)^m} \int_0^1 G^{m,0}_{m,m}\left[\sigma \left| \begin{array}{c} (\gamma_k+\lambda)_1^m \\ (\gamma_k)_1^m \end{array} \right.\right] f(z\sigma^{1/\beta}) d\sigma \quad (22)$$

$$= z^{\beta\lambda} I^{(\gamma_k),(\lambda,\lambda,\ldots,\lambda)}_{(\beta,\beta,\ldots,\beta),m} f(z), \quad m\text{-tuple Erdélyi-Kober integral.}$$

Then our hint was to think about: what was to be if we replace the parameters in the upper row of the above kernel G-function

$$(\gamma_1+\lambda, \gamma_2+\lambda, \ldots, \gamma_m+\lambda) \quad \text{by} \quad (\gamma_1+\delta_1, \gamma_2+\delta_2, \ldots, \gamma_m+\delta_m)$$

with arbitrary and different $\delta_1 > 0, \delta_2 > 0, \ldots, \delta_m > 0$? *This led us to the idea of the operators of fractional integration of multi-order (vector order)* $(\delta_1, \delta_2, \ldots, \delta_m)$, whose theory (GFC) was developed in details in Kiryakova (1986, 1988, 1994).

In Kiryakova (1994, Ch.3): fundamental system solutions of the h-B. diff. equation of m-th order (i.e. of multi-order $(1, 1, ..., 1)$)

$$B\, y(z) = \lambda\, y(z), \quad \lambda \neq 0, \quad 0 < z < \infty, \tag{23}$$

was found. Namely, for $j = 1, ..., m$:

$$y_j(z) = G^{1,0}_{0,m}\left[-\lambda z \middle| \begin{matrix} -- \\ -\gamma_j, -\gamma_1, ..., -\gamma_{j-1}, -\gamma_{j+1}, ..., -\gamma_{m-1}, 0 \end{matrix}\right]$$

$$= \frac{(\lambda t)^{-\gamma_j}}{\prod_{k=1}^{m} \Gamma(\gamma_k + 1)}\ {}_0F_{m-1}\left((1 + \gamma_i - \gamma_j)_{i \neq j}; \lambda z\right) \tag{24}$$

$$:= J^{(m-1)}_{(1+\gamma_i-\gamma_j)_{i \neq j}}(\lambda z),$$ are the hyper-Bessel functions.

These SF are interesting special case of the multi-index M-L functions we shall further consider, as these appear as multi-index Bessel functions !

With respect to relations and use of the kernel H- and G-functions as pdf-functions in probabilty, let us mention the following:

- For $m = 1$ and $m = 2$ these are well known, for example, $G^{m,0}_{m,m}$ for $\sigma < 1$ are (note always $G^{m,0}_{m,m}(\sigma) \equiv 0$ for $|\sigma| > 1$:

$$G^{1,0}_{1,1}\left[\sigma \middle| \begin{array}{c} \gamma+\delta \\ \gamma \end{array}\right] = (1-\sigma)^{\delta-1}\sigma^{\gamma}/\Gamma(\delta),$$ the beta distribution !

$$G^{2,0}_{2,2}\left[\sigma \middle| \begin{array}{c} \gamma_1+\delta_1, \gamma_2+\delta_2 \\ \gamma_1, \gamma_2 \end{array}\right] =$$

$$(1-\sigma)^{\delta_1+\delta_2-1}\sigma^{\gamma_2}/\Gamma(\delta_1+\delta_2)\,{}_2F_1(\gamma_2+\delta_2-\gamma_1, ; \delta_1+\delta_2; 1-\sigma),$$

related to the hypergeometric distribution ... !

Although for $m > 2$ the kernel $G^{m,0}_{m,m}$- and $H^{m,0}_{m,m}$- functions are not among well known but in particular cases can be identified with many SF used in Math. Physics, Engineering, etc.,

an interesting fact is that for arbitrary $m > 1$, the $G^{m,0}_{m,m}$-functions were used by Kabe in statistics as density functions of a random variable, yet long ago:

D.G. Kabe, Some applications of Meijer's G-function to distribution problems in statistics, *Biometrica* **45** (1958), 578-580.

There he also distinguished the cases $m = 1$ and $m = 2$, related to E-K and hypergeometric case. We found also other papers by same author, where SF (especially generalized hypergeometric functions) were used, to express the exact distributions from the known expressions for the moments, for example:

D.G. Kabe, On the exact distribution of a class of multivariate test criteria, *The Annals of Mathematical Statistics* **33**, Number 3 (1962), 1197-1200.

Other particular cases of GFC operators (R-L type)

– Hardy-Littlewood / Cesaro integral operator ($m = 1$);
– Uspensky integral transform ($m = 1$);
– Bessel and Bessel-Clifford differential operators ($m = 2$);
– Poisson and Sonine integral transforms ($m = 1$);
– Poisson-Sonine-Dimovski transmutation operators (arb. $m > 1$);
– Rusheweyh derivative ($m = 1$);
– Weinstein differential operator ($m = 2$);
– Tricomi / Gelersted differntial operators (arb. $m > 1$);
– Ditkin-Prudnikov and Botashev differential operators (arb. $m > 1$):

$$\frac{d}{dz} z \frac{d}{dz} z \ldots \frac{d}{dz} z \frac{d}{dz};$$

– Kraetzel differential operator (arb. $m > 1$);
– In GFT: the operators of Biernacki, Komatu, Libera, Owa-Srivastava, Obradovic, Carlson and Shaffer, Alexander, Hohlov, etc.

Mittag-Leffler type functions – specific SF of FC.

Mittag-Leffler type functions, $\alpha > 0$, $\beta > 0$, $\gamma \in \mathbb{R}$

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = {}_1\Psi_1(z),$$

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k\, z^k}{k!\,\Gamma(\alpha k + \beta)} \quad (Prabhakar \;\; f., \tag{25}$$

are "*fractional index*" ($\alpha > 0$) extensions of the exponential and trigonometric functions $\exp(z)$ and $\cos z$ (when $\alpha = 1$, resp. $\alpha = 2$), satisfying ODEs $D^n y(\lambda z) = \lambda^n y(\lambda z)$ of order $n = 1, 2$.

However, the classical M-L functions (25) and the recently introduced multi-index (vector index) M-L functions (Kiryakova, Luchko et al., Kilbas) as their extensions, appear in the solutions of fractional order problems (of order α, or multi-order $(\alpha_i)_1^m$, $m \geq 1$):

$$E_{(\alpha_i),(\beta_i)}(z) := E^{(m)}_{(\alpha_i),(\beta_i)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha_1 k + \beta_1) \ldots \Gamma(\alpha_m k + \beta_m)}$$

$$= {}_1\Psi_m\left[\begin{array}{c} (1,1) \\ (\beta_i, \alpha_i)_1^m \end{array} \middle| z \right] = H^{1,1}_{1,m+1}\left[-z \,\middle|\, \begin{array}{c} (0,1) \\ (0,1), (1-\beta_i, \alpha_i)_1^m \end{array}\right]. \tag{26}$$

Many of the elementary and special functions are particular cases of $E_{\alpha,\beta}$, and much more – of the multi-index M-L functions (31)-(32).

Examples of the M-L functions:

- $\alpha > 0$, $\beta = 1$: $E_{0,1}(z) = \dfrac{1}{1-z}$; $E_{1,1}(z) = \exp(z)$;
$E_{2,1}(z^2) = \cosh z$, $E_{2,1}(-z^2) = \cos z$; $E_{1/2,1}(z^{1/2}) = \exp(z)\left[1 + \mathrm{erf}(z^{1/2})\right] = \exp(z)\,\mathrm{erfc}(-z^{1/2}) = \exp(z)\left[1 + \dfrac{1}{\sqrt{\pi}}\gamma(\dfrac{1}{2}, z)\right]$ (the error functions, or incomplete gamma functions);
- $\beta \neq 1$: $E_{1,2}(z) = \dfrac{e^z - 1}{z}$; $E_{1/2,2}(z) = \dfrac{\mathrm{sh}\sqrt{z}}{z}$;
$E_{2,2}(z) = \dfrac{\mathrm{sh}\sqrt{z}}{\sqrt{z}}$, etc.
- $\beta = \alpha$: the α-exponential (Rabotnov) function
$y_\alpha(z) = z^{\alpha-1} E_{\alpha,\alpha}(\lambda z^\alpha)$, $\alpha > 0$.

Examples of multi-index M-L functions $E_{(\alpha_i),(\beta_i)}(z)$

• For $\underline{m=1}$ - the classical M-L function $E_{\alpha,\beta}(z)$ with all its special cases (including also the so-called *Rabotnov function*, or α-*exponential function* $y_\alpha(z) = z^{\alpha-1}E_{\alpha,\alpha}(\lambda z)$.

• For $\underline{m=2}$, the function (31) denoted as $\Phi_{\rho_1,\rho_2}(z;\mu_1,\mu_2) = E^{(2)}_{(\frac{1}{\rho}_1,\frac{1}{\rho}_2),(\mu_1,\mu_2)}(z)$, is *Dzrbashjan's M-L type function* (1960).

Some of its particular cases are:

the *M-L function* $E_{\alpha,\beta}(z)$;

the *Bessel function* $J_\nu(z) = (z/2)^\nu E_{(1,1),(\nu+1,1)}(-z^2/4)$;

Struve and Lommel functions $s_{\mu,\nu}(z)$, $H_\nu(z) = const.s_{\nu,\nu}(z)$.

The so-called *Wright function* from the studies of Fox (1928), Wright (1933), Humbert and Agarwal (1953), extended also for $\alpha > -1$, happens is a case of multi-M-L function with $m=2$:

$$\varphi(\alpha,\beta;z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+\beta)} \frac{z^k}{k!} = {}_0\Psi_1\left[\begin{array}{c} - \\ (\beta,\alpha) \end{array}\middle| z\right] = E^{(2)}_{(\alpha,1),(\beta,1)}(z). \tag{27}$$

This Wright f. plays important role in the solution of linear PFDEs as the *fractional diffusion-wave equation* studied by Nigmatullin (1984-1986, to describe the diffusion process in media with fractal geometry, $0 < \alpha < 1$) and by Mainardi et al. (1994-), for propagation of mechanical diffusive waves in viscoelastic media, $1 < \alpha < 2$). In the form $M(z;\beta) = \varphi(-\beta, 1-\beta; -z)$, $\beta := \alpha/2$, it is recently called also as the *Mainardi function*. In our denotations, it is: $M(z;\beta) = E^{(2)}_{(-\beta,1),(1-\beta)}(-z)$ and has examples like: $M(z;1/2) = 1/\sqrt{\pi}\,\exp(-z^2/4)$ and the *Airy function*: $M(z;1/3) = 3^{2/3}\,Ai(z/3^{1/3})$.

In other form and denotation, the Wright function is known as *Wright-Bessel*, or misnamed as *Bessel-Maitland function*:

$$J^{\mu}_{\nu}(z) = \varphi(\mu, \nu+1; -z) = {}_0\Psi_1\left[\begin{array}{c} - \\ (\nu+1,\mu) \end{array}\middle| -z\right]$$
$$= \sum_{k=0}^{\infty} \frac{(-z)^k}{\Gamma(\nu+k\mu+1)\,k!} = E^{(2)}_{(1/\mu,1),(\nu+1,1)}(-z)\,, \qquad (28)$$

again an example of the Dzrbashjan function. It is an obvious *"fractional index" analogue of the classical Bessel function*.

Nowadays, several further *"fractional-indices" generalizations of* $J_\nu(z)$ are exploited, and we can present them as multi-M-L functions! Such one is the so-called *generalized Wright-Bessel (-Lommel) functions*, due to Pathak (1966-1967):

$$J^{\mu}_{\nu,\lambda}(z) = (z/2)^{\nu+2\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{\Gamma(\nu+k\mu+\lambda+1)\Gamma(\lambda+k+1)}$$

$$= (z/2)^{\nu+2\lambda}\, E^{(2)}_{(1/\mu,1),(\nu+\lambda+1,\lambda+1)}\left(-(z/2)^2\right), \ \mu > 0, \tag{29}$$

including, for $\mu = 1$, the *Lommel* (and thus, also the *Struve*) *functions* $J^1_{\nu,\lambda}(z) = \mathrm{const}\, S_{2\lambda+\nu-1,\nu}(z)$. Next one, is the *generalized Lommel-Wright function with 4 indices*, introduced by de Oteiza, Kalla and Conde, $r > 0, n \in \mathbb{N}, \nu, \lambda \in \mathbb{C}$:

$$J^{r,n}_{\nu,\lambda}(z) = (z/2)^{\nu+2\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{k}}{\Gamma(\nu+kr+\lambda+1)\Gamma(\lambda+k+1)^n}$$

$$= (z/2)^{\nu+2\lambda}\, E^{(n+1)}_{(1/r,1,\dots,1),(\nu+\lambda+1,\lambda+1,\dots,\lambda+1)}\left(-(z/2)^2\right). \tag{30}$$

This is an interesting example of a multi-M-L function with $m = n + 1$.

Other special cases:

- For arbitrary $\underline{m \geq 2}$: let $\forall \rho_i = \infty$ $(1/\rho_i = 0)$ and $\forall \mu_i = 1$, $i = 1, ..., m$. Then, from definition of (31),

$$E_{(0,0,...,0),(1,1,...,1)}(z) = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z} \, .$$

- Consider the case $m \geq 2$, $\forall \rho_i = 1$, $i = 1, \ldots, m$, when:

$$E^{(m)}_{(1,1,...,1),(\mu_i+1)}(z) = {}_1\Psi_m \left[\begin{array}{c} (1,1) \\ (\mu_i, 1)_1^m \end{array} \middle| z \right] = \text{const}\, {}_1F_m\,(1; \mu_1, \mu_2, ..., \mu_m; z)$$

reduces to ${}_1F_m$- and to a *Meijer's* $G^{1,1}_{1,m+1}$-function. Denote $\mu_i = \gamma_i + 1, i = 1, \ldots, m$, and let additionally one of the μ_i to be 1, e.g.: $\mu_m = 1$, i.e. $\gamma_m = 0$. Then the multi-M-L function becomes a *hyper-Bessel function* (29) (mentioned in earlier slide):

$$J^{(m-1)}_{\gamma_i,\ldots,\gamma_{m-1}}(z) = \left(\frac{z}{m}\right)^{\sum\limits_{i=1}^{m-1} \gamma_i} E^{(m)}_{(1,1,...,1),(\gamma_1+1,\gamma_2+1,...,\gamma_{m-1}+1,1)} \left(-(\frac{z}{m})^m\right) . \tag{31}$$

In view of the above relation, the multi-index M-L functions with arbitrary $(\alpha_1, \ldots, \alpha_m) \neq (1, \ldots, 1)$ can be seen as *fractional-indices analogues of the hyper-Bessel functions*.

• One may consider *multi-index analogues of the Rabotnov (α-exponential function)*, with all $\mu_i = 1/\rho_i = \alpha > 0, i = 1, \dots, m$:

$$y_\alpha^{(m)}(z) = z^{\alpha-1} E^{(m)}_{(\alpha,..,\alpha),(\alpha,..,\alpha)}(z^\alpha) = z^{\alpha-1} \sum_{k=0}^{\infty} \frac{z^{\alpha k}}{[\Gamma(\alpha+\alpha k)]^m}, \quad (32)$$

$$\alpha = 1: \quad \sum_{k=0}^{\infty} \frac{z^k}{[k!]^m}.$$

• In general, for rational values of $\forall \alpha_i, i = 1, \dots, m$, the multi-index M-L functions (31) are reducible to *Meijer's G-functions* (that is, to classical special functions).

Further *extensions of the multi-M-L functions (31)*, are:

• The *multivariate M-L type functions* by Luchko:

$$E_{(a_1,..,a_n),b}(z_1,..,z_n) := \sum_{k=0}^{\infty} \sum_{\substack{l_1+\cdots+l_n=k \\ l_1 \geq 0,\dots,l_n \geq 0}} (k; l_1, \dots, l_n) \frac{\prod_{i=1}^n z_i^{l_i}}{\Gamma(b + \sum_{i=1}^n a_i l_i)},$$

where $(k; l_1, \dots, l_n)$ are the so-called multinomial coefficients).

• The *M-L type generalization* by Kilbas-Srivastava-Trujillo:

$$E_\rho((\alpha_j, \beta_j)_{1,m}; z) = \sum_{k=0}^{\infty} \frac{(\rho)_k}{\prod_{j=1}^{m} \Gamma(\alpha_j k + \beta_j)} \frac{z^k}{k!}.$$

Basic theory of the multi-index M-L functions (31), see Kiryakova 1999, 2000, 2002, 2010:

Theorem. *The multi-index M-L functions (31) are entire functions of order* ρ *with* $\frac{1}{\rho} = \frac{1}{\rho_1} + \cdots + \frac{1}{\rho_m}$, *and type* $\sigma = \left(\frac{\rho_1}{\rho}\right)^{\frac{\rho}{\rho_1}} \ldots \left(\frac{\rho_m}{\rho}\right)^{\frac{\rho}{\rho_m}} > 1$ *(for* $m > 1$*). Setting* $\mu = \mu_1 + \cdots + \mu_m$, *we have an asymptotic estimate, as:*

$$|E_{(\alpha_i),(\beta_i)}(z)| \leq C|z|^{\rho((1/2)+\mu-(m/2))} \exp\left(\sigma|z|^{\rho}\right), \; |z| \to \infty.$$

We have shown also that the multi-index ML functions are eigen-functions of these G-L operators of multi-order $(\alpha_1, ..., \alpha_m)$:

$$D_{(\alpha_i),(\beta_i)} \, E_{(\alpha_i),(\beta_i)}(\lambda z) = \lambda \, E_{(\alpha_i),(\beta_i)}(\lambda z), \quad \lambda \neq 0.$$

where $D_{(\alpha_i),(\beta_i)}$ is a generalized fractional differentiation operator of the form $z^{-1} \, D^{(\beta_i-\alpha_i-1),(\alpha_i)}_{(1/\alpha_i),m}$ and of multi-order $(1/\alpha_1, ..., 1/\alpha_m)$.

Special Functions of Fractional Calculus as GFC-operators of basic functions

In several our works (Kiryakova, 1994, 1997, 2010, etc.) we have shown the following basic proposition, first for the *classical SF*, because all of them practically are ${}_pF_q$-functions.

Theorem. *All the generalized hypergeometric functions ${}_pF_q$, that is all the classical special functions, can be considered as generalized (q-multiple) fractional integrals (11): $I^{(\gamma_k,\delta_k)}_{(\beta,\dots,\beta),m}$, or / and their respective generalized fractional derivatives $D^{(\gamma_k),(\delta_k)}_{(\beta,\dots,\beta),m}$, of one of the following 3 basic elementary functions, depending on whether $p<q$, $p=q$ or $p=q+1$, resp.:*

$$\cos_{q-p+1}(z)\,(\textit{if } p<q)\,,\; z^{\alpha}\,\exp z\,(\textit{if } p=q)\,,\; z^{\alpha}\,(1-z)^{\beta}\,(\textit{if } p=q+1). \tag{37}$$

Note that these 3 basic functions are also closely related to pdf-functions of known distributions.

Here, the $\cos_m$-function is the *generalized cosine function of m-th order*:

$$y(z) = \cos_m(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{km}}{(km)!},$$

which is the solution of the m-th order IVP

$$y^{(m)}(z) = -y(z); \; y(0) = 1, y'(0) = \cdots = y^{(m-1)}(0) = 0.$$

When $m = 2$, naturally we have the classical $\cos(z) = \cos_2(z)$. Then the corresponding result *for the hyper-Bessel functions* presents our *a generalization of the Poisson integral for the Bessel functions*.

Similar kind of results are also proven for the *Special Functions of Fractional Calculus*, namely for arbitrary Wright generalized hypergeometric functions ${}_p\Psi_q$, by using the operators of GFC with $H^{m,0}_{m,m}$-kernel function, see Kiryakova (2010, *Computers and Math. with Appl.*)

Based on this mentioned result, we have introduced the following **classifications of the classical SF $_pF_q$**, as well as of the SF of FC, represented generally by $_p\Psi_q$:

• If $p < q$, these functions form the class of so-called "trigonometric", or "Bessel"-type SF functions, starting from $_0F_1$, $\cos(z)$ etc.
• If $p = q$, these are the so-called "confluent"-type SF, with simplest example $_1F_1$, $\exp(Z)$, etc.
• If $p = q + 1$, these are the so-called "hypergeometric"-type SF, with simplest example $_2F_1$ (and the geometric series $1/1-z$).

The suitability of such possible classification can be in easier comprehension and interpretation of the SF as somewhat similar to the basic mentioned functions. Note the fractional order integrals may preserve in great extent the asymptotic (in general) and other characteristic behaviour of the treated special or elementary function, so this can be helpful for applied scientists to have idea about.

Images of SF under operators of FC and GFC

Recently many authors are spending lot of time and efforts on the task to evaluate various operators of FC of quite particular SF or of classes of SF. The list of such works, that can be considered also as evaluation improper integrals of products of some elementary and special functions, is rather long and is yet growing daily. Since the special functions present indeed a great variety, and the operators of fractional calculus do as well, the mentioned job produces a huge flood of publications. Most of them use same formal and standard procedures, and besides, often the results sound not of practical use, with except to increase authors' publication activities.

Our approach, based on the ideas of GFC provides a simple and unified way to do such task at once, in the rather general case: both for operators of generalized fractional calculus and for the generalized hypergeometric functions ${}_p\Psi_q$ (as the general SF of FC). Thus, great part of the results in the mentioned publications are well predicted and fall just as particular cases of our general scheme.

See for example,
V. Kiryakova, Fractional calculus operators of special functions? The result is well predictable! *Chaos, Solitons and Fractals* **102** (2017), 2-15.
and other 3 recent publications (*AIP Conf. Proc.* 2017, and to appear: *AIP Conf. Proc.* 2019, *Front. Phys.* 2019). There we have shown that a very large number of papers on the subject include ONLY specification of the results below.

Our extension for E-K operators of Wright g.h.f. ${}_p\Psi_q$

Lemma. *For* $\Re\delta > 0$, $\Re\gamma > -1$, $\mu > 0$, $\lambda \neq 0$, *and if* $p = q + 1$, *we require* $|\lambda z^\mu| < 1$*:*

$$I_\beta^{\gamma,\delta}\left\{{}_p\Psi_q\left[\begin{array}{c}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_q,B_q)\end{array}\middle|\lambda z^\mu\right]\right\}$$
$$= {}_{p+1}\Psi_{q+1}\left[\begin{array}{c}(a_i,A_i)_1^p,(\gamma+1,1/\beta)\\(b_j,B_j)_1^q,(\gamma+\delta+1,1/\beta)\end{array}\middle|\lambda z^\mu\right]. \qquad (33)$$

In particular, $R^\delta\left\{z^{\nu-1}{}_p\Psi_q(\lambda z^\mu)\right\} = z^{\nu+\delta}{}_{p+1}\Psi_{q+1}(\lambda z^\mu)$.

Because, by m-steps: ${}_p\Psi_q \mapsto {}_{p+1}\Psi_{q+1} \mapsto {}_{p+2}\Psi_{q+2} \mapsto \dots$, we have:

Theorem. *For m-tuple $(m = 1, 2, 3, \dots)$ GFC integral with $\delta_k \geq 0, \gamma_k > -1, k = 1, \dots, m$, and $\mu > 0, \lambda \neq 0, c \in \mathbb{R}$,*

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} \left\{ z^c \; {}_p\Psi_q \left[\begin{array}{c} (a_1, A_1), \dots, (a_p, A_p) \\ (b_1, B_1), \dots, (b_q, B_q) \end{array} \middle| \lambda z^\mu \right] \right\}$$

$$= z^c \; {}_{p+m}\Psi_{q+m} \left[\begin{array}{c} (a_i, A_i)_1^p, (\gamma_i + 1 + \frac{c}{\beta_i}, \frac{\mu}{\beta_i})_1^m \\ (b_j, B_j)_1^q, (\gamma_i + \delta_i + 1 + \frac{c}{\beta_i}, \frac{\mu}{\beta_i})_1^m \end{array} \middle| \lambda z^\mu \right]. \quad (34)$$

Corollary. *When all $\beta_k = \beta = 1$, $k = 1, \dots, m$, the image of a ${}_pF_q$ is another function of the same kind with indices increased by the multiplicity m:*

$$I^{(\gamma_k),(\delta_k)}_{1,m} \left\{ {}_pF_q \left(a_1, \dots, a_p; b_1, \dots, b_q; \lambda z \right) \right\}$$

$$= {}_{p+m}F_{q+m} \left(a_1, \dots, a_p, (\gamma_i + 1)_1^m; (b_1, \dots, b_q, (\gamma_i + \delta_i + 1)_1^m; \lambda z \right). \quad (35)$$

Few basic references: (V. Kiryakova)

[1] *Generalized Fractional Calculus and Applications*, Longman–J. Wiley, Harlow–N. York, 1994;

[2] All the special functions are fractional differintegrals of elementary functions. *J. Phys. A: Math. & General* **30** (1997), 5085-5103, doi:10.1088/0305-4470/30/14/019;

[3] Multiple (multiindex) Mittag-Leffler functions and relations to generalized fractional calculus. *J. Comput. Appl. Math.* **118** (2000), 241–259, doi:10.1016/S0377-0427(00)00292-2;

[4] The special functions of fractional calculus as generalized fractional calculus operators of some basic functions. *Computers and Math. with Appl.* **59** (2010), 1128-1141, doi:10.1016/j.camwa.2009.05.014;

[5] The multi-index Mittag-Leffler functions as an important class of special functions of fractional calculus. *Computers and Math. with Appl.* **59** (2010), 1885-1895, doi:10.1016/j.camwa.2009.08.025;

[6] Fractional calculus operators of special functions?–The result is well predictable!. *Chaos Solitons and Fractals* **102** (2017), 2-15, doi:10.1016/j.chaos.2017.03.006.

[7] Use of fractional calculus to evaluate some improper integrals of special functions. *AIP Conf. Proc.* **1910** (2017), doi:10.1063/1.5013994.

[8] Commentary: "A remark on the fractional integral operators and the image formulas of generalized Lommel-Wright function", Front. Phys. (2019), To appear, doi: 10.3389/fphy.2019.00145.