Some inverse problems for time-fractional diffusion equations

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Internationa Workshop on Fractional Calculus
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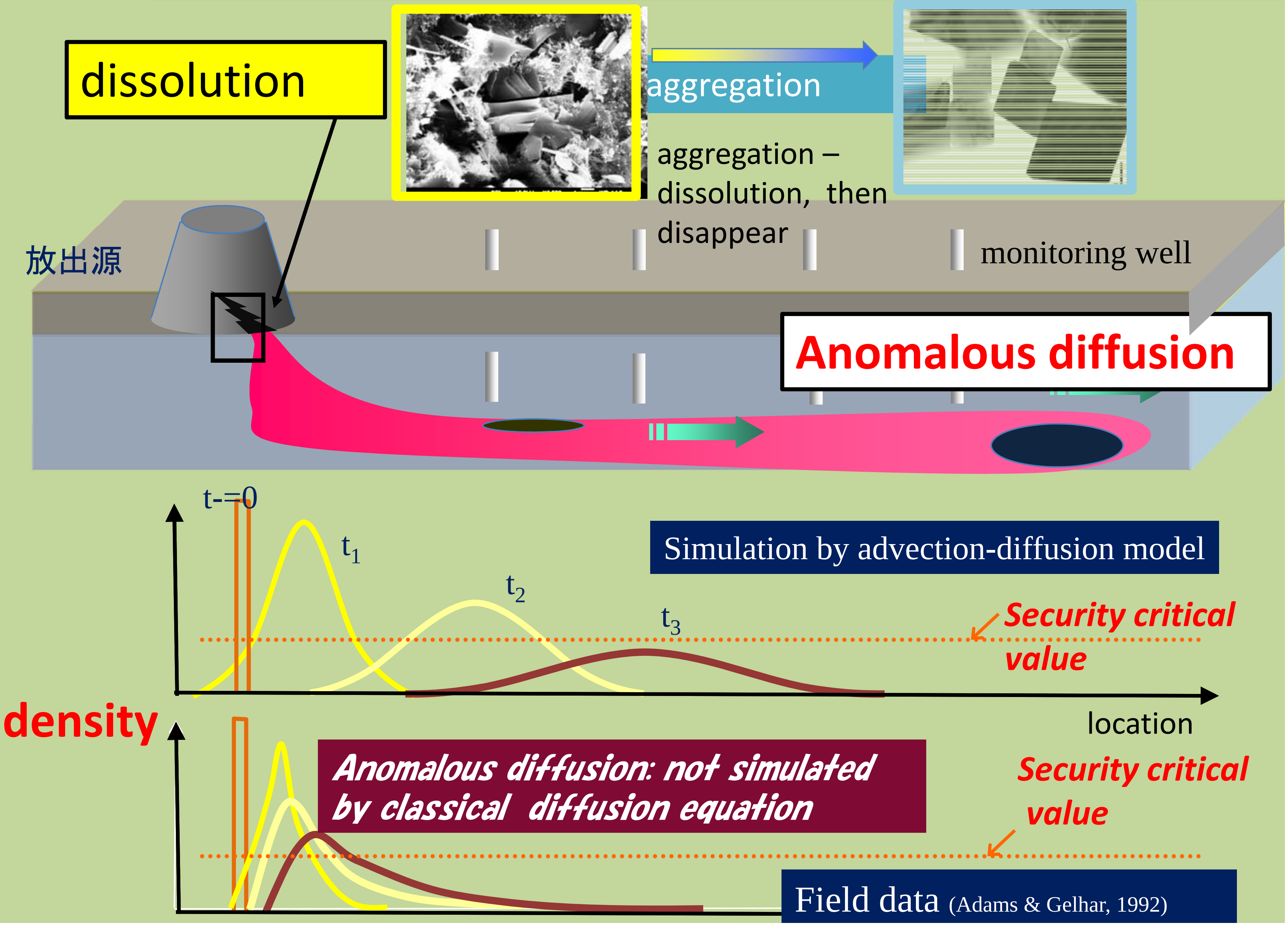
10 June 2020

From ultraslow dissolution to anomalous diffusion



Grand Research Plan

Mission I: Construct general theory for fractional partial differential equations

Launch by Gorenflo-Luchko-Yamamoto 2015

Nonlinear theory, Dynamical system, etc.

Classical theory of PDE

Mission II: Various inverse problems for fractional partial differential equations

Sublime to theory

Optimal control

Parameter identification

motivations

Real world problems: e.g., pollution in soil

Contents:

Three kinds of inverse problems for fractional partial differential equations

Messages:

We have many interesting and important inverse problems.

Contents

§1. Introduction

$\Omega \subset \mathbb{R}^d$ with smooth boundary $\partial\Omega$

$$\partial_t^\alpha u(x,t) = \sum_{i,j=1}^{d} \partial_i(a_{ij}(x)\partial_j u(x,t)) + c(x)u$$

Forward problem:

Solve initial boundary value problem $\Longrightarrow$ prediction

Inverse problems:

How to determine α, a_{ij}, initial, boundary value, shape of Ω. $\Longrightarrow$

modelling, basement for forward problems

Variety of inverse problems!

Mathematical issues for inverse problems: uniqueness, stability

varieties of inverse problems × varieties of fractional equations as models

$\Longrightarrow$ **Varieties^{100}**

Many interesting inverse problems for fractional differential equations!!

§2. Determination of fractional orders

$$u_{\alpha,\beta}\begin{cases} \partial_t^\alpha u = -(-\Delta)^\beta u, & x \in \Omega,\ t > 0, \\ u|_{\partial\Omega} = 0, & t > 0, \\ \begin{cases} u(x,0) = a(x), & x \in \Omega \quad \text{if } 0 < \alpha < 1, \\ u(x,0) = a(x), & \partial_t u(x,0) = 0 \quad \text{if } 1 < \alpha < 2. \end{cases} \end{cases}$$

Inverse problem: determine $\alpha \in (0,2)$ and $\beta \in (0,1)$ by $u(x_0,t)$, $0 < t < T$ with fixed $x_0 \in \Omega$.

Remark.

- Caputo derivative:

$$\partial_t^\alpha v(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \frac{d^n v}{ds^n}(s)ds, \ n-1 < \alpha < n.$$

- We can replace $-\Delta$ by $\sum_{i,j=1}^{d} \partial_i(a_{ij}(x)\partial_j) + c(x)$.
  Maybe we can treat non-symmetric operators.

About forward problems: a monograph

Kubica-Ryszewska-Yamamoto (Springer, 2020)

Remark. $(-\Delta)^{\beta}$: fractional power of $-\Delta$

$\Longleftarrow$ spatial derivative of order 2β

$\alpha, \beta \Longrightarrow$ important physical parameters

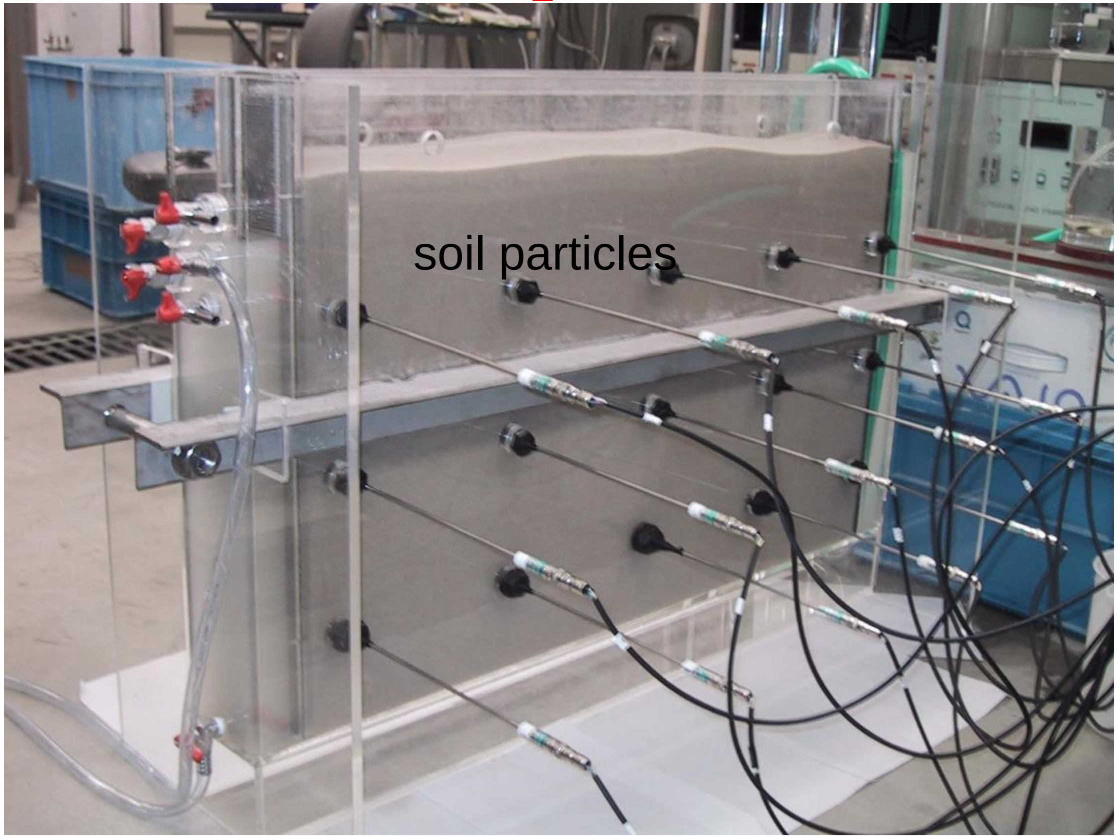
Inverse problem.

Let $x_0 \in \Omega$ be fixed. $u(x_0, t)$, $0 < t < T \Longrightarrow$

$\alpha \in ((0, 2) \setminus \{1\})$ and $\beta \in (0, 1)$.

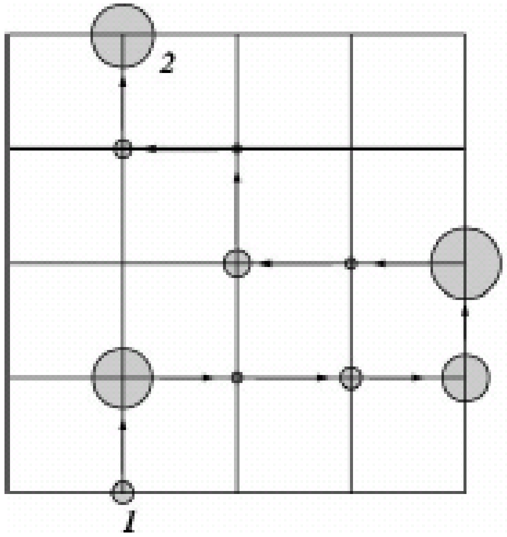
Uniqueness $\Longleftarrow$ How much information data have?

Determination of fractional orders at laboratory based on the micro-model



Prof. Y. Hatano (Tsukuba University)

Comparison of laboratory data with numerical results by Monte Carlo method



Ranom walk

Normal diffusion by Fick's law

$$\langle x^2 \rangle \propto t$$

α=1

Continuous time random walk (micro-model)

$$\langle x^2 \rangle \propto t^{\alpha} \qquad 0 < \alpha < 1$$

Related works on determination of orders

- Hatano, Nakagawa, Wang and Yamamoto 2013
- Li and Yamamoto 2015: uniqueness for mutiterm cases
- Yu, Jing and Qi 2015
- Janno 2016: unique existence
- Janno and Kinash 2018
- Krasnoschok, Pereverzyev, Siryk and Vasylyeva 2019
- Ashurov and Umarov 2020: arXiv:2005.13468v1

Other examples of data:

$$\int_\Omega u(x,t)\rho(x)dx, \quad 0 < t < T: \ \rho\text{: weight}$$

$$\nabla u \cdot \nu(x_0, t), \quad 0 < t < T$$

$\nu(x)$: unit outward normal vector

Preparations

$0 < \lambda_1 < \lambda_2 < \cdots \longrightarrow \infty$: set of eigenvalues of $-\Delta$ with $u|_{\partial\Omega} = 0$.

$-\Delta\varphi_{kj} = \lambda_k\varphi_{kj}$, $1 \le j \le m_k$, m_k: multiplicity of λ_k

φ_{kj}, $1 \le j \le m_k$: orthonormal basis

$(\varphi_{kj}, \varphi_{ij}) := \int_\Omega \varphi_{kj}(x)\varphi_{ki}(x)dx = 1$ if $i = j$, $= 0$ if $i \neq j$

$\lambda_k, \{\varphi_{kj}\}_{1 \le j \le m_k}$: eigensystem of $-\Delta$ with $u|_{\partial\Omega} = 0$

Fractional power of $-\Delta$:

$$(-\Delta)^{\beta} v = \sum_{k=1}^{\infty} \lambda_k^{\beta} \sum_{j=1}^{m_k} (v, \varphi_{kj}) \varphi_{kj},$$

$v \in D((-\Delta)^{\beta}) \subset H^{2\beta}(\Omega)$: Sobolev-Slobodeckij space

$$(-\Delta)^{-\beta} a = \frac{\sin \pi\beta}{\pi} \int_0^{\infty} \eta^{-\beta} (-\Delta + \eta)^{-1} a \, d\eta$$

in $L^2(\Omega)$ with $0 < \beta < 1$ (e.g., Pazy).

$$E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} :$$ Mittag-Leffler function

Theorem 1 (uniqueness).

Let $a \in H_0^2(\Omega)$ *if* $d = 1, 2, 3$ *(*$a \in H_0^{2\gamma}(\Omega)$ *with* $\gamma > \frac{d}{4}$*),*

$$a \geq 0, \not\equiv 0 \quad \text{or} \quad \leq 0, \not\equiv 0 \quad \text{in } \Omega,$$

and

$$\exists k_0 \in \mathbb{N}, \sum_{j=1}^{m_{k_0}} (a, \varphi_{k_0 j}) \varphi_{k_0 j}(x_0) \neq 0, \quad \lambda_{k_0} \neq 1.$$

Then $u_{\alpha,\beta}(x_0, t)$, $0 < t < T$

$\Longleftrightarrow (\alpha, \beta) \in (0, 2) \setminus \{1\} \times (0, 1)$ *1 to 1*

Simplified case.

Assume $m_k = 1$: λ_k is simple for all k, $-\Delta\varphi_k = \lambda_k\varphi_k$.

Corollary (uniqueness).

(i) $a \in H_0^2(\Omega)$ *if* $d = 1, 2, 3$,

$$a \geq 0, \not\equiv 0 \quad \text{or} \quad \leq 0, \not\equiv 0 \quad \text{in } \Omega$$

(ii) $\exists k_0 \in \mathbb{N}$, $(a, \varphi_{k_0})\varphi_{k_0}(x_0) \neq 0$ *and* $\lambda_{k_0} \neq 1$.

Then $u_{\alpha,\beta}(x_0, t)$, $0 < t < T \Longleftrightarrow$

$(\alpha, \beta) \in ((0, 2) \setminus \{1\}) \times (0, 1)$ *1 to 1*

Remark.

- (ii) is essential for uniqueness of β.
  Let $\lambda_1 = 1$ and $-\Delta\varphi_1 = \varphi_1$. Then
  $u_{\alpha,\beta}(x,t) = E_{\alpha,1}(-t^\alpha)\varphi_1(x)$ for $x \in \Omega,\ t > 0$.
  No information of β!
- (i) $\Longrightarrow$ uniqueness for α
- Tatar-Ulusoy (2013):
  $(a, \varphi_k) > 0$ for all $k \Longrightarrow$ uniqueness.
  Our proof producces the same conclusion.
- Let $\lambda_k \neq 1$ for all k.
  Then $a(x_0) \neq 0 \Longrightarrow$ uniqueness for β.

Main ingredients for proof.

(1) Eigenfunction expansion

(2) Asymptotics of Mittag-Leflller function

$$E_{\alpha,1}(-t) = \sum_{\ell=1}^{N} \frac{(-1)^{\ell+1}}{\Gamma(1-\alpha\ell)} \frac{1}{t^{\ell}} + O\left(\frac{1}{t^{1+N}}\right), \quad t > 0, \to \infty$$

(3) Strong maximum principle for $(-\Delta)^{\beta}$:

Lemma (strong maximum principle for $(-\Delta)^\beta$):

$a \geq 0, \not\equiv 0, a|_{\partial\Omega} = 0 \Longrightarrow (-\Delta)^\beta a(x) > 0, x \in \Omega$

Proof:

$\beta = 1 \Longrightarrow$ classical strong maximum principle

$0 < \beta < 1$: strong maximum principle $\Longrightarrow$

$(-\Delta + \eta)^{-1} a > 0$ in $\Omega \Longrightarrow$

$$((-\Delta)^{-\beta})a(x) = \frac{\sin \pi\beta}{\pi} \int_0^\infty \eta^{-\beta}(-\Delta + \eta)^{-1} a \, d\eta > 0$$

■

Proof for simple eigenvalues

$$u_{\alpha,\beta}(x_0,t) = \sum_{k=1}^{\infty} E_{\alpha,1}(-\lambda_k^{\beta} t^{\alpha})(a,\varphi_k)\varphi_k(x_0)$$

$$= \frac{1}{\Gamma(1-\alpha)} \frac{1}{t^{\alpha}} \sum_{k=1}^{\infty} \frac{(a,\varphi_k)\varphi_k(x_0)}{\lambda_k^{\beta}} + O\left(\frac{1}{t^{2\alpha}}\right)$$

$$= \frac{1}{\Gamma(1-\alpha)} \frac{1}{t^{\alpha}} (-\Delta)^{\beta} a(x_0) + O\left(\frac{1}{t^{2\alpha}}\right) \quad \text{as } t \to \infty$$

$u_{\alpha,\beta}(x_0,t) = u_{\alpha_1,\beta_1}(x_0,t)$, $0 < t < T \Longrightarrow$

$$\frac{1}{\Gamma(1-\alpha)} \frac{1}{t^{\alpha}} (-\Delta)^{\beta} a(x_0) + O\left(\frac{1}{t^{2\alpha}}\right) = \frac{1}{\Gamma(1-\alpha_1)} \frac{1}{t^{\alpha_1}} (-\Delta)^{\beta_1} a(x_0) + O\left(\frac{1}{t^{2\alpha_1}}\right)$$

as $t \to \infty \Longrightarrow$

$(-\Delta)^{\beta} a(x_0) \neq 0$, $(-\Delta)^{\beta_1} a(x_0) \neq 0$ by $x_0 \in \Omega \Longrightarrow \alpha = \alpha_1$

Proof of $\beta = \beta_1$.

Set $a_k = (a, \varphi_k)\varphi_k(x_0)$.

$$\sum_{\ell=1}^{N} \frac{(-1)^{\ell+1}}{\Gamma(1-\alpha\ell)t^{\alpha\ell}} \sum_{k=1}^{\infty} \frac{a_k}{\lambda_k^{\beta\ell}} = \sum_{\ell=1}^{N} \frac{(-1)^{\ell+1}}{\Gamma(1-\alpha\ell)t^{\alpha\ell}} \sum_{k=1}^{\infty} \frac{a_k}{\lambda_k^{\beta_1\ell}} + O\left(\frac{1}{t^{\alpha(1+N)}}\right)$$

as $t \to \infty \Longrightarrow$

$$\sum_{k=1}^{\infty} \frac{a_k}{\lambda_k^{\beta\ell}} = \sum_{k=1}^{\infty} \frac{a_k}{\lambda_k^{\beta_1\ell}}, \quad \ell \in \mathbb{N} \setminus \{m/\alpha\}_{m\in\mathbb{N}}$$

Let $k_0 \in \mathbb{N}$ such that $a_1 = \cdots = a_{k_0-1} = 0$ and $a_{k_0} \neq 0$, $\lambda_{k_0} \neq 1$.

Assume $\lambda_{k_0} > 1$ (similar for other case): Let $\beta_1 > \beta \Longrightarrow$

$$a_{k_0}\left(1 - (\lambda_{k_0}^{\beta-\beta_1})^{\ell}\right) + \sum_{k=k_0+1}^{\infty} a_k \left((\lambda_{k_0}^{\beta}/\lambda_k^{\beta})^{\ell} - (\lambda_{k_0}^{\beta}/\lambda_k^{\beta_1})^{\ell}\right) = 0$$

for $\ell \in \mathbb{N} \setminus \{m/\alpha\}_{m\in\mathbb{N}}$.

$$\left|\lambda_{k_0}^{\beta-\beta_1}\right|, \quad \left|\lambda_{k_0}^{\beta}/\lambda_k^{\beta}\right|, \quad \left|\lambda_{k_0}^{\beta}/\lambda_k^{\beta_1}\right| < 1$$

for $k \geq k_0 + 1$.

Choose $\ell_n \in \mathbb{N} \setminus \{m/\alpha\}_{m\in\mathbb{N}}$ such that $\lim_{n\to\infty} \ell_n = \infty$

$\Longrightarrow a_{k_0} = 0$: contradiction

§3. Fractional diffusion equations backward in time

with Profs G. Floridia (Università Mediterranea di Reggio Calabria) and Z. Li (Shandong University of Technology)

$-Au := \sum_{i,j=1}^{n} \partial_i(a_{ij}(x)\partial_j u) + c(x)u$,

where $c \le 0$, $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$.

$$\begin{cases} \partial_t^\alpha u = -Au \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad u(\cdot, T) = b. \end{cases}$$

unique existence

$u \in C([0,T]; L^2(\Omega)) \cap C((0,T]; H^2(\Omega) \cap H_0^1(\Omega))$

to $\partial_t^\alpha u = -Au$ with $u|_{\partial\Omega} = 0$ and $u(\cdot, T) = b$

(Sakamoto-Yamamoto 2011)

- memory effect or weak smoothing
- Different from $\alpha = 1$.

Preliminaries

(I) $0 < \lambda_1 \le \lambda_2 \le ....$: eignvalues of A with multiplicities,
φ_k: eigenfunction for λ_k, $(\varphi_j, \varphi_k) = \delta_{jk}$. $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$

(II) $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}$ (Mittag-Leffler function).

(III) $S(t)a = \sum_{k=1}^{\infty} (a, \varphi_k) E_{\alpha,1}(-\lambda_k t^\alpha)\varphi_k$,
$K(t)a = \sum_{k=1}^{\infty} t^{\alpha-1}(a, \varphi_k) E_{\alpha,\alpha}(-\lambda_k t^\alpha)\varphi_k, \quad a \in L^2(\Omega)$

$\Longrightarrow$

$$\|S(t)a\|_{H^2(\Omega)} \le Ct^{-\alpha}\|a\|, \quad \|S(t)a\| \le C\|a\|$$

$\|A^\gamma K(t)a\| \le Ct^{\alpha(1-\gamma)-1}\|a\|$, $0 \le \gamma \le 1$, $t > 0$

(IV)

$$\begin{cases} \partial_t^\alpha u = -Au + F \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad u(\cdot, 0) = a. \end{cases}$$

$\Longrightarrow$

$$u(t) = S(t)a + \int_0^t K(t-s)F(s)ds, \quad t > 0.$$

(V) $S(t) : L^2(\Omega) \longrightarrow H^2(\Omega) \cap H_0^1(\Omega)$ is isomorphism for $t > 0$.

Sakamoto-Yamamoto (2011):

$$-Lu := \sum_{i,j=1}^{n} \partial_i(a_{ij}(x)\partial_j u) + \sum_{j=1}^{n} b_j(x)\partial_j u(x) + c(x)u$$

$$=: -Au + Bu$$

$$\begin{cases} \partial_t^\alpha u = -Lu \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad u(T) = b. \end{cases}$$

Theorem 3.

For $b \in H^2(\Omega) \cap H_0^1(\Omega)$, there exists unique solution u with same regularity as in Theorem 2.

Key to Proof

First Step.

$\partial_t^\alpha u = -Au + Bu$ in Ω, $u|_{\partial\Omega} = 0$, $u(\cdot, 0) = a \Longrightarrow$

$u_a(t) = S(t)a + \int_0^t K(t-s)Bu_a(s)ds$, where $S(t)$, $K(t)$ are constructed for $-A \Longrightarrow b = S(T)a + \int_0^T K(T-s)Bu_a(s)ds \Longleftrightarrow$

$$a = S(T)^{-1}\left(b - \int_0^T K(T-s)Bu_a(s)ds\right)$$

$$= S(T)^{-1}b - S(T)^{-1}\int_0^T K(T-s)Bu_a(s)ds = S(T)^{-1}b - Ma$$

Here

$$Ma := S(T)^{-1}\int_0^T K(T-s)Bu_a(s)ds$$

Unique solvability for a?

Second Step: compactness of $M : L^2(\Omega) \longrightarrow L^2(\Omega)$.

We can prove $\|Au(t)\| \le Ct^{-\alpha}\|a\|$ (Sakamoto-Yamamoto for $B = 0$)

Let $0 < \delta < \frac{1}{2}$.

$$\left\|A^{1+\delta}\int_0^T K(T-s)Bu_a(s)ds\right\| = \left\|\int_0^T A^{\frac{1}{2}+\delta}K(T-s)A^{\frac{1}{2}}Bu_a(s)ds\right\|$$

$$\le C\int_0^T (T-s)^{\alpha(\frac{1}{2}-\delta)-1}\|Au_a(s)\|ds \le C\int_0^T (T-s)^{\alpha(\frac{1}{2}-\delta)-1}s^{-\alpha}ds\|a\| \le C\|a\|.$$

$\Longrightarrow$

$$\left\|A^{\delta}S(T)^{-1}\int_0^T K(T-s)Bu_a(s)ds\right\| \le C\left\|A^{1+\delta}\int_0^T K(T-s)Bu_a(s)ds\right\| \le C\|a\|.$$

$\Longrightarrow M : L^2(\Omega) \longrightarrow H^{2\delta}(\Omega)$ is bounded

$M : L^2(\Omega) \longrightarrow L^2(\Omega)$ is compact

Third Step:

Fredholm equation of second kind:

$a = S(T)^{-1}b - Ma$

Well-posedness $\Longleftarrow$ "$b = 0 \Longrightarrow a = 0$"

$\Longleftrightarrow$ Backward uniqueness

$$\begin{cases} \partial_t^\alpha u = -Lu \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad u(\cdot, T) = 0 \end{cases} \Longrightarrow \quad u(\cdot, 0) = 0.$$

Fourth Step: Backward uniqueness.

P_m, $m \in \mathbb{N}$: eigenprojection for eigenvalue λ_m of L

Closed subspace spanned by all the generalized eigenfunctions is $L^2(\Omega)$. (Agmon)

$\Longrightarrow$ $P_m a = 0$, $m \in \mathbb{N}$ imply $a = 0$.

Set

$$u_m(t) := P_m u(t), \quad a_m := P_m a$$

$\Longrightarrow$

$\partial_t^\alpha u_m = (-\lambda_m + D_m)u_m =: J u_m$ (Jordan form) where $D_m^\ell = 0$ with large ℓ.

$$\Longrightarrow \qquad u_m(t) = E_{\alpha,1}(Jt^\alpha)a_m = \sum_{k=0}^{\infty} \frac{(Jt^\alpha)^k}{\Gamma(\alpha k + 1)} a_m$$

Set $u_m = (u_m^1, ..., u_m^N)^T$ and $a_m = (a_m^1, ..., a_m^N)^T$ where ^{T}: tranpose
We can represent

$$u_m^1(t) = E_{\alpha,1}(-\lambda_m t^\alpha)a_m^1 + \cdots$$

$$u_m^2(t) = E_{\alpha,1}(-\lambda_m t^\alpha)a_m^2 + \cdots$$

$$\cdots\cdots$$

$$u_m^N(t) = E_{\alpha,1}(-\lambda_m t^\alpha)a_m^N$$

$u(T) = 0 \Longrightarrow u_m(T) = 0$

By $E_{\alpha,1}(-\lambda_m T^\alpha) \neq 0 \Longrightarrow$
$a_m^N = 0$, then
$a_m^{N-1} = 0$, then .....
$a_m^1 = 0$.
$\Longrightarrow a_m = 0$ for $m \in \mathbb{N}$.

Thus $a = 0$!

§4. Inverse coefficient problems

Let $0 < \alpha < 1$, $\Omega \subset \mathbb{R}^d$: bounded,

$\omega \subset \Omega$: arbitrarily fixed subdomain.

$$u_p(x,t) \begin{cases} \partial_t^\alpha u = \Delta u + p(x)u(x,t), \quad x \in \Omega \subset \mathbb{R}^d, \\ \partial_\nu u|_{\partial\Omega} = 0, \quad u(x,0) = a(x), \quad x \in \Omega \end{cases}$$

Inverse coefficient problem.

$u_p|_{\omega\times(0,T)} \Longrightarrow p(x)$, $x \in \Omega$.

Similar result with data $\partial_\nu u|_{\Gamma\times(0,T)}$ with $\Gamma \subset \partial\Omega$.

Limited references:

(1) Dirichlet-to-Neumann map:

Imanuvilov-Li-Yamamoto (2016),

Kian-Oksanen-Soccorsi-Yamamoto (2018), etc.

(2) $\partial_t^{1/2} u - \partial_x^2$: 1D

Yamamoto-Zhang (2012)

(3) Integral transform to wave equation:

Miller-Yamamoto (2013)

(4) Spectral approach for 1D case:

Cheng-Nakagawa-Yamamoto-Yamazaki (2009),

Li-Zhang-Jia-Yamamoto (2013), etc.

Li-Zhang-Jia-Yamamoto (2013):

$\varphi_k''(x) + p(x)\varphi_k(x) = -\lambda_k\varphi_k,\ \varphi_k'(0) = \varphi_k'(1) = 0$

$\psi_k''(x) + q(x)\psi_k(x) = -\mu_k\psi_k,\ \psi_k'(0) = \psi_k'(1) = 0$ for $k \in \mathbb{N}$

$\Longrightarrow$

Let $a \in H_0^4(0,1)$ *and* $(a, \varphi_k) \neq 0$ *for all* $k \in \mathbb{N}$. *If* $u_p(i,t) = u_q(i,t)$ *for* $0 < t < T$ *and* $i = 0, 1$, *then* $p(x) = q(x)$ *for* $0 < x < 1$.

Remark. We can determine α uniquely.

Scenario of proof:

Data of u = [eigenfunction expansion] $\Longrightarrow$ Spectral data = [Gel'fand-Levitan theory] $\Longrightarrow$ Uniqueness

Step 1:

(1) 1D case: well by $\lambda_n = c_0 n + O(1/n)$

(2) General dimensions: bottleneck because only $\lambda_n = c_0 n^{2/d} + o(n^{2/d})$.

Step 2:

Multidimensional Gel'fand-Levitan theory (e.g., Nachman-Sylvester-Uhlmann)

Class of initial values for uniqueness.

A: $\{k \in \mathbb{N};\ (a, \varphi_k) \ne 0\} = \mathbb{N}$.

Uniqueness for 1D case, ??? for multidimensions

B: $^{\#}\{k \in \mathbb{N};\ (a, \varphi_k) \ne 0\} = \infty$.

Uniqueness ????

C: $^{\#}\{k \in \mathbb{N};\ (a, \varphi_k) \ne 0\} < \infty$.

Uniqueness holds. $\Longleftarrow$ We discuss here!

λ_k: eigenvalue of $-\Delta - p(x)$ with $\partial_\nu u|_{\partial\Omega} = 0$
P_k: orthogonal projection onto Ker $(-\lambda_k - (\Delta + p))$
μ_k: eigenvalue of $-\Delta - q(x)$ with $\partial_\nu u|_{\partial\Omega} = 0$
Q_k: orthogonal projection onto Ker $(-\mu_k - (\Delta + q))$

$$u_p \begin{cases} \partial_t^\alpha u = \Delta u + p(x)u, \quad x \in \Omega \subset \mathbb{R}^d,\ t > 0, \\ \partial_\nu u|_{\partial\Omega} = 0, \quad u(x,0) = a(x). \end{cases}$$

Inverse problem:
$\omega \subset \Omega$: arbitraily given subdomain
Determine p in Ω by $u_p|_{\omega\times(0,T)}$.

Theorem 4 (Jing-Yamamoto: 2020).

Let $p, q, a \in H^\infty(\Omega)$,

$$^{\#}\{k \in \mathbb{N};\ P_k a \not\equiv 0 \quad \text{in } \Omega\} < \infty$$

and $|a(x)| > 0$, $x \in \Omega$.

Then $u_p = u_q$ *in* $\omega \times (0, T)$ *implies* $p = q$ *in* Ω.

Remark.

(i) The set of a for the uniqueness is dense (not very special choice).

(ii) $^{\#}\{k \in \mathbb{N};\ P_k a \not\equiv 0 \quad \text{in } \Omega\} = \mathbb{N} \Longrightarrow$ Uniqueness for 1D case

Thank you very much.