Smoothing properties of semigroups generated by accretive quadratic operators

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Accretive quadratic operators

Given $q : \mathbb{R}^{2n} \to \mathbb{C}$ a complex-valued quadratic form with a non-negative real-part, we consider the accretive quadratic operator $q^w(x, D_x)$ defined by the Weyl quantization of the symbol q

$$q^w(x, D_x)u = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle} q\left(\frac{x+y}{2}, \xi\right) u(y)\ \mathrm{d}y\mathrm{d}\xi,$$

and equipped with the domain

$$D(q^w) = \big\{u \in L^2(\mathbb{R}^n) : q^w(x, D_x)u \in L^2(\mathbb{R}^n)\big\}.$$

The operator $q^w(x, D_x)$ is a non-selfadjoint differential operator, since

$$\left(x^\alpha \xi^\beta\right)^w = \frac{1}{2}\left(x^\alpha D_x^\beta + D_x^\beta x^\alpha\right),$$

for all $(\alpha, \beta) \in \mathbb{N}^{2n}$ such that $|\alpha| + |\beta| = 2$, with $D_x = -i\partial_x$.

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Smoothing directions of the phase space

Problematic :

Understand how the possible non-commutation phenomena between the selfadjoint and the skew-selfadjoint parts of the operator q^w allow the evolution operators $e^{-tq^w} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ to enjoy smoothing properties in specific directions of the phase space.

More precisely, we aim at describing the vector subspaces $\Sigma \subset \mathbb{R}^{2n}$ of the phase space satisfying that for all $t > 0$, $m \geq 1$, $X_1, \ldots, X_m \in \Sigma$, there exists a positive constant $C_{t,m,X_1,\ldots,X_m} > 0$ such that for all $u \in L^2(\mathbb{R}^n)$,

$$\left\| \langle X_1, X\rangle^w \ldots \langle X_m, X\rangle^w e^{-tq^w} u \right\|_{L^2(\mathbb{R}^n)} \leq C_{t,m,X_1,\ldots,X_m} \|u\|_{L^2(\mathbb{R}^n)},$$

and to sharply describe the dependence of the constant $C_{t,m,X_1,\ldots,X_m}$ with respect to $t > 0$, $m \geq 1$ and $X_1, \ldots, X_m \in \Sigma$.

Notation :

For all $X_0 = (x_0, \xi_0) \in \mathbb{R}^{2n}$, we denote by $\langle X_0, X\rangle^w$ the following differential operator

$$\langle X_0, X\rangle^w = \langle x_0, x\rangle + \langle \xi_0, D_x\rangle,$$

where $\langle \cdot, \cdot \rangle$ stands for the canonical Euclidean scalar product of $\mathbb{R}^{2n}$.

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Singular space and smoothing properties

The singular space $S \subset \mathbb{R}^{2n}$ of the quadratic form q, introduced in [Hitrik & Pravda-Starov 09], is the vector subspace of the phase space defined by the following intersection of kernels

$$S = \bigcap_{j=0}^{2n-1} \operatorname{Ker}(\operatorname{Re} F(\operatorname{Im} F)^j) \subset \mathbb{R}^{2n},$$

where $F = JQ$ is the Hamilton map of the quadratic form q.

Theorem (A. & Bernier 2019):

There exist some positive constants $c > 1$ and $t_0 > 0$ such that for all $m \geq 1$, $X_1, \dots, X_m \in S^{\perp}$, $0 < t < t_0$ and $u \in L^2(\mathbb{R}^n)$,

$$\left\| \langle X_1, X \rangle^w \dots \langle X_m, X \rangle^w e^{-tq^w} u \right\|_{L^2(\mathbb{R}^n)} \leq \frac{c^m}{t^{k_{X_1} + \dots + k_{X_m} + \frac{m}{2}}} \left(\prod_{j=1}^{m} |X_j| \right) \sqrt{m!}\, \|u\|_{L^2(\mathbb{R}^n)},$$

where $0 \leq k_{X_j} \leq k_0$ denotes the index of the vector $X_j \in S^{\perp}$.

The integer $0 \leq k_0 \leq 2n - 1$ is a structural parameter of the singular space S.

The notion of index is linked to the structure of the space $S^{\perp}$.

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A polar decomposition result

Theorem (A. & Bernier 2019):

There exists a family $(a_t)_{t\in\mathbb{R}}$ of non-negative quadratic forms $a_t : \mathbb{R}^{2n} \to \mathbb{R}_+$ that depend analytically on the time-variable $t \in \mathbb{R}$ and a family $(U_t)_{t\in\mathbb{R}}$ of metaplectic operators such that

$$\forall t \geq 0, \quad e^{-tq^w} = e^{-ta_t^w} U_t, \quad \text{with} \quad e^{-ta_t^w} = e^{-sa_t^w}\big|_{s=t}.$$

Moreover, there exist some positive constants $c > 0$ et $T > 0$ such that for all $0 \leq t \leq T$ and $X \in \mathbb{R}^{2n}$,

$$a_t(X) \geq c \sum_{j=0}^{k_0} t^{2j} \operatorname{Re} q\big((\operatorname{Im} F)^j X\big).$$

There also exists a family $(b_t)_{-T<t<T}$ of real quadratic forms $b_t : \mathbb{R}^{2n} \to \mathbb{R}$ that depend analytically on the time-variable $-T < t < T$, such that

$$\forall t \in [0, T), \quad e^{-tq^w} = e^{-ta_t^w} e^{-itb_t^w}, \quad \text{with} \quad e^{-itb_t^w} = e^{-isb_t^w}\big|_{s=t}.$$

Sketch of proof:

Use the Fourier integral operator structure of the evolution operators e^{-tq^w}.

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Remark:

The same study can be performed for semigroups generated by fractional Ornstein-Uhlenbeck operators

$$P = \frac{1}{2}\langle QD_x, D_x\rangle^s + \langle Bx, \nabla_x\rangle, \quad x \in \mathbb{R}^n,$$

with B and Q some real $n \times n$ matrices, Q being symmetric positive semidefinite, and $s > 0$ a positive real number.

Applications:

1. Establish subelliptic estimates enjoyed by the quadratic operator q^w.
2. Tackle null-controllability issues for the evolution equation

$$\begin{cases} \partial_t f(t,x) + q^w(x, D_x) f(t,x) = h(t,x)\mathbb{1}_\omega(x), \quad t > 0,\ x \in \mathbb{R}^n, \\ f(0,\cdot) = f_0 \in L^2(\mathbb{R}^n), \end{cases}$$

with $\omega \subset \mathbb{R}^n$ a Borel set with a positive Lebesgue measure.

Another class of operators and applications

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