Hardy and Hardy-Sobolev inequalities on Heisenberg groups

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Abstract

This work [3] is devoted to present the geometric Hardy and Hardy-Sobolev inequalities for the sub-Laplacian in the half-spaces of the Heisenberg group with a sharp constant. This result answers a conjecture posed by S. Larson in [2]. As a consequence, a geometric Hardy-Sobolev-Maz'ya inequality is recovered.

Preliminaries on the Heisenberg group:

Let $\mathbb{H}^n$ be the Heisenberg group, that is, the set $\mathbb{R}^{2n+1}$ equipped with the group law

$$\xi \circ \tilde{\xi} := (x+\tilde{x}, y+\tilde{y}, t+\tilde{t}+2\sum_{i=1}^{n}(\tilde{x}_i y_i - x_i\tilde{y}_i)),$$

where $\xi := (x,y,t) \in \mathbb{H}^n$, $x := (x_1,\ldots,x_n)$, $y := (y_1,\ldots,y_n)$, and $\xi^{-1} = -\xi$ is the inverse element of ξ with respect to the group law. The dilation operation of the Heisenberg group with respect to the group law has the following form $\delta_\lambda(\xi) := (\lambda x, \lambda y, \lambda^2 t)$ for $\lambda > 0$.
The Lie algebra $\mathfrak{h}$ of the left-invariant vector fields on the Heisenberg group $\mathbb{H}^n$ is spanned by

$$X_i := \frac{\partial}{\partial x_i} + 2y_i\frac{\partial}{\partial t} \text{ and } Y_i := \frac{\partial}{\partial y_i} - 2x_i\frac{\partial}{\partial t}$$

and with their (non-zero) commutator $[X_i, Y_i] = -4\frac{\partial}{\partial t}$. The horizontal gradient of $\mathbb{H}^n$ is $\nabla_H := (X_1,\ldots,X_n,Y_1,\ldots,Y_n)$
Let us define the half-space of the Heisenberg group by

$$\mathbb{H}^+ := \{\xi \in \mathbb{H}^n : \langle \xi, \nu\rangle > d\},$$

where $\nu := (\nu_x, \nu_y, \nu_t)$ with $\nu_x, \nu_y \in \mathbb{R}^n$ and $\nu_t \in \mathbb{R}$ is the Riemannian outer unit normal to $\partial\mathbb{H}^+$ (see [1]) and $d \in \mathbb{R}$. The Euclidean distance to the boundary $\partial\mathbb{H}^+$ is defined by

$$dist(\xi, \partial\mathbb{H}^+) := \langle \xi, \nu\rangle - d.$$

Let us define

$$X_i(\xi) = (\underbrace{0,\ldots,\overset{i}{1},\ldots,0}_{n}, \underbrace{0,\ldots,0}_{n}, 2y_i),$$

$$Y_i(\xi) = (\underbrace{0,\ldots,0}_{n}, \underbrace{0,\ldots,\overset{i}{1},\ldots,0}_{n}, -2x_i).$$

Then we have

$$\langle X_i(\xi), \nu\rangle = \nu_{x,i} + 2y_i\nu_t, \qquad \langle Y_i(\xi), \nu\rangle = \nu_{y,i} - 2x_i\nu_t,$$

where $\xi := (x,y,t)$ with $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$, $\nu := (\nu_x, \nu_y, \nu_t)$.

Introduction

The Hardy inequality in the half-space on the Heisenberg group was shown by Luan and Young in [4] as follows

$$\int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi \geq \int_{\mathbb{H}^+} \frac{|x|^2+|y|^2}{t^2}|u|^2 d\xi. \tag{1}$$

An alternative proof of this inequality was given by Larson in [2], where the author generalised it to any half-space of the Heisenberg group,

$$\int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi \geq \frac{1}{4}\int_{\mathbb{H}^+} \frac{\sum_{i=1}^n \langle X_i(\xi),\nu\rangle^2 + \langle Y_i(\xi),\nu\rangle^2}{dist(\xi,\partial\mathbb{H}^+)^2}|u|^2 d\xi,$$

where X_i and Y_i (for $i = 1,\ldots,n$) are left-invariant vector fields on the Heisenberg group, ν is the Riemannian outer unit normal to the boundary. Also, there is the L^p-generalisation of the above inequality

$$\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^+} \frac{\sum_{i=1}^n |\langle X_i(\xi),\nu\rangle|^p + |\langle Y_i(\xi),\nu\rangle|^p}{dist(\xi,\partial\mathbb{H}^+)^p}|u|^p d\xi. \tag{2}$$

Conjecture posed by S. Larson

A more natural weight in the right-hand side of (2) would be

$$\frac{\left(\sum_{i=1}^n \langle X_i(\xi),\nu\rangle^2 + \langle Y_i(\xi),\nu\rangle^2\right)^{p/2}}{dist(\xi,\partial\mathbb{H}^+)^p}.$$

L^p-Hardy inequality on $\mathbb{H}^+$

Let $\mathbb{H}^+$ be a half-space of the Heisenberg group $\mathbb{H}^n$. Then for all functions $u \in C_0^\infty(\mathbb{H}^+)$ and $p > 1$ we have

$$\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi \geq C\int_{\mathbb{H}^+} \frac{\left(\sum_{i=1}^n \langle X_i(\xi),\nu\rangle^2 + \langle Y_i(\xi),\nu\rangle^2\right)^{p/2}}{dist(\xi,\partial\mathbb{H}^+)^p}|u|^p d\xi, \tag{3}$$

where the constant $C := \left(\frac{p-1}{p}\right)^p$ is sharp.

L^2-Hardy inequality on $\mathbb{H}^+$

Let $\mathbb{H}^+$ be a half-space of the Heisenberg group $\mathbb{H}^n$. Then for all functions $u \in C_0^\infty(\mathbb{H}^+)$ we have

$$\int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi \geq \int_{\mathbb{H}^+} \frac{|x|^2+|y|^2}{t^2}|u|^2 d\xi, \tag{4}$$

where the constant is sharp.
This corollary can be proved by considering $p = 2$, $dist(\xi,\partial\mathbb{H}^+) = t$ and $\nu = (0,0,1)$.

Hardy-Sobolev inequality on $\mathbb{H}^+$

Let $\mathbb{H}^+$ be a half-space of the Heisenberg group $\mathbb{H}^n$. Then for every function $u \in C_0^\infty(\mathbb{H}^+)$ and $2 \leq p < Q$ with $Q = 2n+1$, there exists some $C_1 > 0$ such that we have

$$\left(\int_{\mathbb{H}^+} |\nabla_H u|^p d\xi - C\int_{\mathbb{H}^+} \frac{\left(\sum_{i=1}^n \langle X_i(\xi),\nu\rangle^2 + \langle Y_i(\xi),\nu\rangle^2\right)^{p/2}}{dist(\xi,\partial\mathbb{H}^+)^p}|u|^p d\xi\right)^{\frac{1}{p}} \geq C_1\left(\int_{\mathbb{H}^+} |u|^{p^*} d\xi\right)^{\frac{1}{p^*}},$$

where $p^* := Qp/(Q-p)$ and the constant $C := \left(\frac{p-1}{p}\right)^p$.

Hardy-Sobolev-Maz'ya inequality

Let $\mathbb{H}^+$ be a half-space of the Heisenberg group $\mathbb{H}^n$. Then for every function $u \in C_0^\infty(\mathbb{H}^+)$, there exists some $C > 0$ such that we have

$$\left(\int_{\mathbb{H}^+} |\nabla_H u|^2 d\xi - \int_{\mathbb{H}^+} \frac{|x|^2+|y|^2}{t^2}|u|^2 d\xi\right)^{\frac{1}{2}} \geq C\left(\int_{\mathbb{H}^+} |u|^{2^*} d\xi\right)^{\frac{1}{2^*}},$$

where $2^* := 2Q/(Q-2)$ with the homogeneous dimension $Q = 2n+1$.

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