Discrete Pseudo-differential Operators and Boundary Value Problems in Hypercomplex Function Theory

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Outline

From Continuous to Discrete - Principal approaches

- Sampling theorems and spectral decomposition of Laplacian on combinatorial graphs (Pesenson)
- Moving to "higher-dimensional" graphs (quad-graphs, diamond-shaped graphs): finite differences (Forgy/Schreiber)
- DeRham complexes on meshes, Finite Element exterior calculus (Arnold, Christensen, Desbrun)
- Discrete differential geometry: simplicial complexes, (Bobenko)

Common ground: Creation of a graded algebra (discrete differential forms)

From Continuous to Discrete - Clifford Analysis

- First approach in Clifford analysis: Gürlebeck/Sproessig, operator theory/potential theory;

  Based on Kirchhoff laws, one has the concept of a *potential function* satisfying to the *discrete star Laplacian*.

$$\Delta_h f(hk) := \frac{1}{h^2}\left[\sum_{i=1}^{n}[f(hk + h\mathbf{e}_i) + f(hk - h\mathbf{e}_i)] - 2n\, f(hk)\right],$$

  for $k \in \mathbb{Z}^n$.

- Recently: C., Faustino, Kähler, Ku, de Ridder, Sommen (Dirac operator on upper half lattice).

Forward and backward differences

The forward and backward differences $\partial_h^{\pm j}$ are given by

$$\partial_h^{+j} f(hm) = \frac{1}{h}[f(hm + h\mathbf{e}_j) - f(hm)],$$

$$\partial_h^{-j} f(hm) = -\frac{1}{h}[f(hm) - f(hm - h\mathbf{e}_j)]$$

whit lattice constant $h > 0$ and $m = m_1\mathbf{e}_1 + \cdots + m_n\mathbf{e}_n \in \mathbb{Z}^n$.

Split the basis element $\mathbf{e}_j$ ($j = 1, \cdots, n$) into two basis elements

$$\mathbf{e}_j = \mathbf{e}_j^+ + \mathbf{e}_j^-$$

corresponding to the forward and backward directions.

Factorization of the star-Laplacian

- The elements $\mathbf{e}_j^+, \mathbf{e}_j^-$, $(j = 1, \cdots, n)$ form a Witt basis for the complexified Clifford algebra $\mathbb{C}_n$, satisfying to:

$$\begin{cases} \mathbf{e}_j^- \mathbf{e}_k^- + \mathbf{e}_k^- \mathbf{e}_j^- & = & 0, \\ \mathbf{e}_j^+ \mathbf{e}_k^+ + \mathbf{e}_k^+ \mathbf{e}_j^+ & = & 0, \\ \mathbf{e}_j^+ \mathbf{e}_k^- + \mathbf{e}_k^- \mathbf{e}_j^+ & = & -\delta_{jk}, \end{cases}$$

  with δ_{jk} the Kronecker symbol.

- Discrete Dirac operator $D_h^{+-} = \sum_{j=1}^n \mathbf{e}_j^+ \partial_h^{+j} + \mathbf{e}_j^- \partial_h^{-j}$ which factorizes the star-Laplacian as

$$\Delta_h =: \sum_{j=1}^n \partial_h^{+j} \partial_h^{-j} = -(D_h^{+-})^2,$$

- Its adjoint Dirac operator

$$D_h^{-+} = \sum_{j=1}^n \mathbf{e}_j^+ \partial_h^{-j} + \mathbf{e}_j^- \partial_h^{+j}.$$

Discrete Fourier Transform

- ℓ^p spaces $(1 \leq p < +\infty)$

$$u \in \ell^p(\Omega) \quad iff \quad \|u\|_{\ell^p(\Omega)} := \left(\sum_{hm\in\Omega} |u(hm)|^p h^3\right)^{1/p} < \infty$$

- Discrete Fourier transform $\mathcal{F}_h : \ell^2(\mathbb{Z}^n) \to L_0^2\left(\left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^n\right)$,

$$u \mapsto \mathcal{F}_h u(\xi) = \begin{cases} \sum_{m\in\mathbb{Z}^n} e^{ihm\cdot\xi} u(hm)h^n, & \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^n \\ 0 & \text{other } \xi \end{cases}$$

- Inverse $\mathcal{F}_h^{-1} = R_h\mathcal{F}$ where R_h is the restriction to the lattice $h\mathbb{Z}^n$ and

$$\mathcal{F}f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(\xi)\chi_{\left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^n}(\xi)d\xi, \quad x \in \mathbb{R}^n.$$

Discrete Fundamental Solutions

Discrete fundamental solution of D_h^{-+}

$$D_h^{-+}E_h^{-+}(hm)=\delta_h(hm)=\begin{cases} h^{-n} & m=0 \\ 0 & m\neq 0 \end{cases}, \quad \forall m\in\mathbb{Z}^n$$

where δ_h is the discrete Dirac function.

Remark that the symbol d^2 of the discrete star Laplacian is known

$$\mathcal{F}_h(-\Delta_h u)(\xi)=\frac{4}{h^2}\sum_{j=1}^{n}\sin^2\left(\frac{\xi_j h}{2}\right)\mathcal{F}_h u(\xi):=d^2\mathcal{F}_h u(\xi)$$

and that

$$\mathcal{F}_h(D_h^{-+}u)(\xi):=\tilde{\xi}_-\mathcal{F}_h u(\xi)=\sum_{j=1}^{n}\left(\mathbf{e}_j^+\xi_{-j}^D+\mathbf{e}_j^-\xi_{+j}^D\right)\mathcal{F}_h u(\xi),$$

with $\xi_{\pm j}^D=\mp h^{-1}\left(1-e^{\mp ih\xi_j}\right)$.

Computation in Fourier Domain - 1

These leads to

$$E_h^{-+} = R_h \mathcal{F}\left(\frac{\widetilde{\xi}_-}{d^2}\right) = \sum_{j=1}^{n} \mathbf{e}_j^+ R_h \mathcal{F}\left(\frac{\xi_{-j}^D}{d^2}\right) + \mathbf{e}_j^- R_h \mathcal{F}\left(\frac{\xi_{+j}^D}{d^2}\right),$$

with

- $D_h^{-+} E_h^{-+}(hm) = \delta_h(hm), \quad hm \in \ h\mathbb{Z}^n,$
- $\lim_{h\to 0} \frac{\widetilde{\xi}_-}{d^2} = \frac{-i\xi}{|\xi|^2};$
- $\left| \int_{[-\frac{\pi}{h},\frac{\pi}{h}]^n} \frac{\xi_{\pm j}^D}{d^2} e^{-i\langle hm,\xi\rangle} d\xi \right| \approx \mathcal{O}(\frac{1}{|hm|^{n-1}}),$

so that $E_h^{-+} \in \ell_p(\mathbb{Z}^n, \mathbb{C}_n)$ for $p > \frac{n}{n-1}$.

Computation in Fourier Domain - 2

Lemma

For each fix point $m \in h\mathbb{Z}^n$ of the lattice, there exists a constant $C > 0$ such that

$$\left|E_h^{-+}(m) - E(m)\right| \leq C \frac{h}{|m|^n}, \quad m \neq \mathbf{0},$$

where E is the fundamental solution of the Dirac operator in $\mathbb{R}^n$.

Similarly for $E_h^{+-} = R_h \mathcal{F}\left(\frac{\widetilde{\xi}_+}{d^2}\right) = \sum_{j=1}^n \mathbf{e}_j^+ R_h \mathcal{F}\left(\frac{\xi_{+j}^D}{d^2}\right) + \mathbf{e}_j^- R_h \mathcal{F}\left(\frac{\xi_{-j}^D}{d^2}\right)$.

Discrete Borel-Pompeiu Formulae

From the discrete Stokes formula and translates of the fundamental solution we obtain

Lemma (Upper Discrete Borel-Pompeiu Formula)

For $f \in \ell^p(\mathbb{Z}^{n-1} \times \mathbb{Z}_+; \mathbb{C}_n)$, $1 \leq p < +\infty$, we have

$$\sum_{\underline{\eta} \in \mathbb{Z}^n} \left[E_h^{-+}(\underline{\eta} - \underline{m}, -m_n)\mathbf{e}_n^+ f(\underline{\eta}, 1) + E_h^{-+}(\underline{\eta} - \underline{m}, 1 - m_n)\mathbf{e}_n^- f(\underline{\eta}, 0) \right] h^n$$

$$+ \sum_{\eta \in \mathbb{Z}^n \times \mathbb{Z}_+} E_h^{-+}(\eta - m) \left[D_h^{+-} f(\eta) \right] h^n = \begin{cases} 0, & \text{if } m \notin \mathbb{Z}^{n-1} \times \mathbb{Z}_+, \\ -f(m), & \text{if } m \in \mathbb{Z}^{n-1} \times \mathbb{Z}_+. \end{cases}$$

where $m = (\underline{m}, m_n)$.

Similar lemma holds for the lower half Lattice $\mathbb{Z}^{n-1} \times \mathbb{Z}_-$.

Discrete Cauchy Transforms

Definition (Upper Cauchy transform)

For a discrete ℓ^p-function f, $1 \leq p < +\infty$, defined on the boundary layers $(\underline{\eta}, 0), (\underline{\eta}, 1), \underline{\eta} \in \mathbb{Z}^{n-1}$, we define the upper Cauchy transform for $m \in \mathbb{Z}^{n-1} \times \mathbb{Z}_+$ as

$$C^+[f](m) = -\sum_{\underline{\eta} \in \mathbb{Z}^{n-1}} \left[E_h^{-+}(\underline{\eta} - \underline{m}, -m_n)\mathbf{e}_n^+ f(\underline{\eta}, 1) \right. \\ \left. + E_h^{-+}(\underline{\eta} - \underline{m}, 1 - m_n)\mathbf{e}_n^- f(\underline{\eta}, 0) \right] h^{n-1}.$$

Definition (Lower Cauchy Transform)

For a discrete ℓ^p-function f, $1 \leq p < +\infty$, defined on the boundary layers $(\underline{\eta}, -1), (\underline{\eta}, 0), \underline{\eta} \in \mathbb{Z}^{n-1}$, we define the lower Cauchy transform for $m \in \mathbb{Z}^{n-1} \times \mathbb{Z}_-$ as

$$C^-[f](m) = \sum_{\underline{\eta} \in \mathbb{Z}^{n-1}} \left[E_h^{-+}(\underline{\eta} - \underline{m}, -1 - m_n)\mathbf{e}_n^+ f(\underline{\eta}, 0) + \right. \\ \left. + E_h^{-+}(\underline{\eta} - \underline{m}, -m_n)\mathbf{e}_n^- f(\underline{\eta}, -1) \right] h^{n-1}.$$

Discrete upper / lower Hilbert transform

Theorem (C./Kähler/Ku/Sommen 2014)

Let $f \in \ell^p(\mathbb{Z}^{n-1}, \mathbb{C}_n)$ be a boundary value of a discrete monogenic function in the upper (lower) half space. Then its $(n-1)$D-Fourier transform $F = \mathcal{F}_h f$ satisfies

$$\pm\frac{\widetilde{\xi}_-}{\underline{d}}\left(\mathbf{e}_n^{\mp}\frac{2}{h\underline{d}-\sqrt{4+h^2\underline{d}^2}}+\mathbf{e}_n^{\pm}\frac{h\underline{d}-\sqrt{4+h^2\underline{d}^2}}{2}\right)F=F.$$

Definition (Discrete upper (lower) Hilbert transform)

$$H_\pm f = \mathcal{F}_h^{-1}\left[\pm\frac{\widetilde{\xi}_-}{\underline{d}}\left(\mathbf{e}_n^{\mp}\frac{2}{h\underline{d}-\sqrt{4+h^2\underline{d}^2}}+\mathbf{e}_n^{\pm}\frac{h\underline{d}-\sqrt{4+h^2\underline{d}^2}}{2}\right)\right]\mathcal{F}_h f.$$

Moreover

$$(H_+)^2 = id = (H_-)^2$$

Discrete Riesz transforms (arbitrary nD domains)

The discrete Riesz transforms are obtained by discrete convolution with the Fourier kernels

$$R_i^+ := \mathbf{\Phi}^{-1}\mathcal{F}_h^{-1}\left[\frac{\widetilde{\xi}_{-,i}}{\underline{d}}\left(\mathbf{e}_i^+\frac{h\underline{d}-\sqrt{4+h^2\underline{d}^2}}{2}+\mathbf{e}_i^-\frac{2}{h\underline{d}-\sqrt{4+h^2\underline{d}^2}}\right)\right],$$

and

$$R_i^- := \mathbf{\Phi}^{-1}\mathcal{F}_h^{-1}\left[\frac{\widetilde{\xi}_{-,i}}{\underline{d}}\left(\mathbf{e}_i^+\frac{2}{h\underline{d}-\sqrt{4+h^2\underline{d}^2}}+\mathbf{e}_i^-\frac{h\underline{d}-\sqrt{4-h^2\underline{d}^2}}{2}\right)\right],$$

where $\mathbf{\Phi}$ denotes a mapping of individual boundary parts to the real line.

Plemelj-Sokhotzki Formulae

Due to the properties of H_+ and H_- we can introduce the projectors into the respective Hardy spaces

Definition

$$P_+ = \frac{1}{2}(I + H_+), \quad Q_+ = \frac{1}{2}(I - H_+).$$
$$P_- = \frac{1}{2}(I + H_-), \quad Q_- = \frac{1}{2}(I - H_-).$$

Lemma

$$f \in h_p^+ \quad \text{iff} \quad P_+ f = f; \qquad\qquad f \in h_p^- \quad \text{iff} \quad P_- f = f.$$

Discrete half Dirichlet problems

Problem I

Given $g \in \ell^p(\mathbb{Z}^{n-1})(1 \leq p < \infty)$ we want to find $f : \mathbb{Z}^{n-1} \times \mathbb{Z}_0^+ \to \mathbb{C}_n$ such that

$$\begin{cases} D_h^{-+} f(hm) = 0, & m \in \mathbb{Z}^{n-1} \times \mathbb{Z}^+ \\ f(h(\underline{m},1)) = g(h\underline{m}), & \underline{m} \in \mathbb{Z}^{n-1}. \end{cases} \qquad (1)$$

Theorem

Problem I is uniquely solvable if and only if $g \in h_p^+$.
Moreover, its solution is given by $f = C^+[\mathcal{G}_+ g]$ in $\mathbb{Z}_+^n$.

Idea of proof: $g \notin h_p^+$ implies it doesn't exist a discrete monogenic f such that its $1-$ layer fulfils the given boundary condition. So, $g \in h_p^+$.

Now, $f = C^+[\mathcal{G}_+ g]$ is a discrete monogenic function on the upper half lattice with $f(h(\underline{m},1)) = g(\underline{m})$ and uniqueness ensured by the maximum principle.

Jump problem

Problem II

Given $g \in \ell^p(h\mathbb{Z}^{n-1}), (1 \leq p < \infty)$ determine a discrete function $f : h\mathbb{Z}^n \to \mathbb{C}_n$ subjected to a jump condition,

$$\begin{cases} D_h^{-+} f(hm) = 0, & m \in \mathbb{Z}^n \setminus \{m_n = 0\} \\ \mathbf{e}_n^- f_+(h\underline{m}) - \mathbf{e}_n^+ f_-(h\underline{m}) = \mathbf{e}_n g(h\underline{m}), & \underline{m} \in \mathbb{Z}^{n-1} \end{cases} . \qquad (2)$$

Theorem

The Riemann boundary value problem (2) is uniquely solvable, and its solution is explicitly given by

$$f(hm) = \begin{cases} C^+ \mathcal{G}_+[g_+](hm) & , m_n \geq +1 \\ C^- \mathcal{G}_-[-g_-](hm) & , m_n \leq -1 \end{cases} ,$$

where $g_- := g^1 + \mathbf{e}_n^- g^3,\ g_+ := g^1 - g^4 + \mathbf{e}_n^+ g^2$.

Convergence Result - Discrete to Continuous

- $g \in L_p(\mathbb{R}^{n-1}, \mathbb{C}_n) \cap C^\alpha(\mathbb{R}^{n-1}, \mathbb{C}_n)$ $(0 < \alpha < 1)$, and $1 < p < \infty$, then we have

$$\|R_h g\|_{\ell^{p+\frac{n-1}{\alpha}}} \leq C\|g\|_{L_p};$$

- The continuous Cauchy transform

$$C_\Gamma f(y) = \int_{\mathbb{R}^{n-1}} E(x-y)(-\mathbf{e}_n) f(x) d\Gamma_x, \quad y = (\underline{y}, y_n) \in \mathbb{R}^n, y_n > 0,$$

with $E(x)$ being the fundamental solution to the Dirac operator $D = \sum_{j=1}^n \mathbf{e}_j \frac{\partial}{\partial_{x_j}}$

Theorem (C./Kähler/Ku 2015)

Let $g \in H_p^+ \cap C^\alpha\left(\mathbb{R}^{n-1}, \mathbb{C}_n\right) \cap W_p^1\left(\mathbb{R}^{n-1}, \mathbb{C}_n\right)$, for $0 < \alpha \leq 1$, and $1 < p < n$. Then, the following estimate for the point-wise error between the discrete solution of (1) and its continuous counterpart holds:

$$\left|C_\Gamma[g](mh) - C^+\mathcal{G}_+[g](mh)\right| \leq \left(Ah^{n-1} + Bh\right)\|g\|_{L^p}, \quad (3)$$

for all $m \in \mathbb{Z}_+^n$, with $A, B > 0$ constants independent of h and g.

ΨDO's on $\mathbb{R}^n$ (*Kohn/Nirenberg, Hörmander*)

Definition [pseudo-differential operator (ΨDO)]

We say that

$$Af(x) = \int_{\mathbb{R}^n} \sigma_A(x,\xi) e^{2\pi i x\cdot\xi}\, \hat{f}(\xi) d\xi.$$

is a pseudo-differential operator with symbol $\sigma_A(x,\xi)$.

Definition [symbol class $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$]

We say that $\sigma_A = \sigma_A(x,\xi) \in S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ if

1. $\sigma_A = \sigma_A(x,\xi)$ is smooth on $\mathbb{R}^n \times \mathbb{R}^n$;
2. for all $x,\xi \in \mathbb{R}^n$ and all $\alpha,\beta \in \mathbb{N}_0^n$ there exists a constant $C_{\alpha,\beta}$ such that

$$\left|\partial_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi)\right| \leq C_{\alpha,\beta} \langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}, \quad \langle\xi\rangle := \left(1+|\xi|^2\right)^{1/2}.$$

Then

1. $\sigma_A \in S^m(\mathbb{R}^n \times \mathbb{R}^n) \quad \Leftrightarrow \quad A \in \Psi^m(\mathbb{R}^n \times \mathbb{R}^n)$;
2. $L = \sum_{|\alpha|\leq m} a_\alpha(x)\partial_x^\alpha$ has symbol $\sigma_L(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)(2\pi i\xi)^\alpha$.
3. we write $A = \sigma_A(x,D)$.

The torus case $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$

Agranovich 1979, Ruzhansky/Turunen/Wirth since 2006

- For $\hat{f} : \mathbb{Z}^n \to \mathbb{C}$ we define the difference operator Δ_{ξ_j} as

$$\Delta_{\xi_j}\hat{f} = \hat{f}(\xi + \mathbf{e}_j) - \hat{f}(\xi), \quad j = 1, \cdots, n,$$

  and

$$\Delta_\xi^\alpha \hat{f} = \Delta_{\xi_1}^{\alpha_1} \cdots \Delta_{\xi_n}^{\alpha_n} \hat{f}, \quad \alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n.$$

- For periodic $f : \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{C}$ we have

$$\hat{f}(\xi) = \int_{\mathbb{T}^n} e^{-2\pi i x\cdot\xi} f(x)dx, \quad \xi \in \mathbb{Z}^n.$$

  Then,

$$Af(x) = \sum_{\xi\in\mathbb{Z}^n} \sigma(x,\xi) e^{2\pi i x\cdot\xi}\ \hat{f}(\xi),$$

  with symbol class $S^m_{1,0}(\mathbb{R}^n \times \mathbb{Z}^n)$ defined as

$$\left|\Delta_\xi^\alpha \partial_x^\beta \sigma(x,\xi)\right| \leq C_{\alpha,\beta} \left(1 + |\xi|^2\right)^{(m-|\alpha|)/2}$$

Discrete symbols (*Botchway / Kibiti / Ruzhansky, 2020*)

Let $\sigma : \mathbb{Z}^n \times \mathbb{T}^n \to \mathbb{C}$ be measurable.

- We define the pseudodifferential operator acting on $\mathbb{Z}^n$ and associated to σ as
$$Op(\sigma)f(x) := \int_{\mathbb{T}^n} e^{2\pi i x\cdot\xi}\sigma(x,\xi)\hat{f}(\xi)d\xi, \quad x \in \mathbb{Z}^n.$$
- The discrete Schwartz space $\mathcal{S}(\mathbb{Z}^n)$ (space of rapidly decreasing function) is the space of $\varphi : \mathbb{Z}^n \to \mathbb{C}$ if for any $M > 0$ there exists $C_{\varphi,M} > 0$ such that
$$|\varphi(x)| \leq C_{\varphi,M}(1 + |x|^2)^{-M/2}, \quad \forall x \in \mathbb{Z}^n,$$
endowed with the topology induced by the seminorms
$$p_j(\varphi) := \sup_{x\in\mathbb{Z}^n}(1 + |x|^2)^{j/2}|\varphi(x)|.$$
- The space of tempered distributions $\mathcal{S}'(\mathbb{Z}^n)$ is the topological dual to $\mathcal{S}(\mathbb{Z}^n)$.

Symbol class

Definition (symbol class $S^m_{\rho,\delta}(\mathbb{Z}^n \times \mathbb{T}^n)$)

We say that $\sigma : \mathbb{Z}^n \times \mathbb{T}^n \to \mathbb{C}$ belongs to the class $S^m_{\rho,\delta}(\mathbb{Z}^n \times \mathbb{T}^n)$ if

- $\sigma(x,\cdot) \in C^\infty(\mathbb{T}^n)$ for all $x \in \mathbb{Z}^n$;
- for all $\alpha, \beta \in \mathbb{N}_0^n$ there exists $C_{\alpha,\beta} > 0$ s. t.

$$\left|\partial_\xi^\beta \Delta_x^\alpha \sigma(x,\xi)\right| \leq C_{\alpha,\beta}(1+|x|^2)^{\frac{1}{2}(m-\rho|\alpha|+\delta|\beta|)},$$

for all $x \in \mathbb{Z}^n, \xi \in \mathbb{T}^n$.

Of particular interest is the class $S^m_{1,0}$

$$\left|\partial_\xi^\beta \Delta_x^\alpha \sigma(x,\xi)\right| \leq C_{\alpha,\beta}(1+|x|^2)^{\frac{1}{2}(m-|\alpha|)},$$

thus corresponding to controlled decay in "space"-variable.

Spectral theory for Clifford valued operators

Problems:

- due to the non-commutativity of the Clifford setting we can only work on right-Clifford modules $\mathcal{H} \otimes \mathcal{C}\ell_{0,n}$ with

$$\langle\langle x, y \rangle\rangle = \sum \langle x_A, y_B \rangle_{\mathcal{H}} \mathbf{e}_A \overline{\mathbf{e}}_B, \qquad \langle x, y \rangle = [\langle\langle x, y \rangle\rangle]_0;$$

- for a bounded complex linear operator A acting on a Banach space $\mathcal{B}$

$$\sigma(A) = \{\lambda \in \mathbb{C} : \ \lambda \mathrm{I} - A \text{ is not invertible}\}.$$

But for a $\mathcal{C}\ell_{0,n}$−valued operator T

$$Tv = sv \qquad \text{(right linear)}$$

while

$$Tv = vs \qquad \text{(non-linear)}.$$

Furthermore, the “eigenvalue” s acts as an equivalence class.

Quaternionic case: S-spectrum

We are interested on left linear operators since

$$-(T^2 - 2s_0 T + |s|^2) \sum_{n=0}^{\infty} T^n s^{-1-n} = T - \overline{s}$$

In praxis,

$$(\lambda \mathrm{I} - A)^{-1} \quad \rightsquigarrow \quad -(T^2 - 2s_0 T + |s|^2)^{-1}(T - \overline{s}).$$

For a bounded quaternionic left linear operator T we have

$$\text{S-spectrum} \quad \sigma_S(T) := \{s \in \mathbb{H} : T^2 - 2s_0 T + |s|^2 \text{ is invertible}\}.$$

The pseudo-resolvent operator

$$Q_S(T) := (T^2 - 2s_0 T + |s|^2)^{-1}$$

is defined on the S-resolvent

$$\rho_S(T) = \mathbb{H} \setminus \sigma_S(T).$$

Clifford pseudo-differential operator

Definition

We say that

$$Tf(x) = \int_{\mathbb{T}^n} \sigma_T(x,\xi) e^{2\pi i x\cdot\xi} \hat{f}(\xi) d\xi,$$

is a pseudo-differential operator with symbol $\sigma_T(x,\xi)$ with values in $\mathcal{C}\ell_{0,n}$.

We say that $\sigma_T = \sigma_T(x,\xi) \in S^m_{\rho,\delta}(\mathbb{Z}^n \times \mathbb{T}^n)$ if

1. $\sigma_T = \sigma_T(x,\xi)$ is smooth on $\mathbb{Z}^n \times \mathbb{T}^n$;
2. for all $x, \xi \in \mathbb{Z}^n$ and all $\alpha, \beta \in \mathbb{N}_0^n$ there exists a constant $C_{\alpha,\beta}$ such that

$$\left|\partial_\xi^\alpha (D_h^{+-})_x^\beta \sigma_T(x,\xi)\right| \leq C_{\alpha,\beta} \langle x \rangle^{m-\rho|\alpha|+\delta|\beta|}, \quad \langle x \rangle := \left(1+|x|^2\right)^{1/2}.$$

Special case: $\rho = 1, \delta = 0$.

Convolutional kernel representations

For a measurable symbol $\sigma : \mathbb{Z}^n \times \mathbb{T}^n \to \mathbb{C}_n$ we define the pseudo-difference operator

$$f \in \ell^2(\mathbb{Z}^n) \to Op(\sigma)f(x) := \int_{\mathbb{T}^n} e^{2\pi i x\cdot\xi}\sigma(x,\xi)\hat{f}(\xi)d\xi,$$

and the quantization $\sigma \mapsto Op(\sigma)$ is called the lattice quantization.

These operators have a (convolutional) kernel representation

$$Op(\sigma) = \int_{\mathbb{T}^n} e^{2\pi i x\cdot\xi}\sigma(x,\xi)\hat{f}(\xi)d\xi = \int_{\mathbb{T}^n} e^{2\pi i x\cdot\xi}\sigma(x,\xi)\sum_{s\in\mathbb{Z}^n} e^{-2\pi i s\cdot\xi}f(s)d\xi$$

$$= \sum_{m\in\mathbb{Z}^n}\left(\int_{\mathbb{T}^n} e^{2\pi i m\cdot\xi}\sigma(x,\xi)d\xi\right)f(x-m) = \sum_{m\in\mathbb{Z}^n}\kappa(x,m)f(x-m).$$

Ellipticity

Definition

We say $\sigma \in S^m_{1,0}(\mathbb{Z}^n \times \mathbb{T}^n)$ is elliptic of order m if it exits constants $C, M > 0$ such that

$$|\sigma(x,\xi)| \geq C(1+|x|^2)^{m/2},$$

for all $\xi \in \mathbb{T}^n$ and $x \in \mathbb{Z}^n : |x| \geq M$.

But the discrete star Laplacian

$$\Delta_h f := \sum_{i=1}^{n} [f(\cdot + \mathbf{e}_i) + f(\cdot - \mathbf{e}_i)] - 2n\, f$$

has symbol

$$\sigma(x,\xi) = \frac{1}{2\sum_{j=1}^{n} \cos(2\pi\xi_j) - 2n} = \frac{1}{4\sum_{j=1}^{n} \sin^2(\pi\xi_j)}$$

and, hence it is **not** a discrete elliptic operator.

Boundedness on $\ell^2(\mathbb{Z}^n)$

Mikhlin type of boundedness on $\ell^2(\mathbb{Z}^n)$

Let $N \in \mathbb{N}$ such that $N > n/2$. Suppose that the symbol $\sigma : \mathbb{Z}^n \times \mathbb{T}^n \mapsto \mathbb{R}_{0,n}$ satisfies

$$|\partial_\xi^\beta \sigma(x,\xi)| \leq C, \qquad \text{for all } (x,\xi) \in \mathbb{Z}^n \times \mathbb{T}^n, \text{ and all } |\beta| \leq N.$$

Then $Op(\sigma)$ extends to a bounded operator on $\ell^2(\mathbb{Z}^n)$.

For $s \in \mathbb{R}$ and $1 \leq p < \infty$ we consider the weighted space $\ell_p^s(\mathbb{Z}^n)$ via

$$\|f\|_{\ell_p^s(\mathbb{Z}^n)} := \left(\sum_{x \in \mathbb{Z}^n} (1 + |x|^2)^{sp/2} |f(x)|^p \right)^{1/p} < \infty$$

Weighted boundedness on $\ell^2(\mathbb{Z}^n)$

Let $m \in \mathbb{R}$ and $\sigma \in S_{0,0}^m(\mathbb{Z}^n \times \mathbb{T}^n)$. Then $Op(\sigma)$ is bounded from $\ell_2^s(\mathbb{Z}^n)$ to $\ell_2^{s-m}(\mathbb{Z}^n)$ for all $s \in \mathbb{R}$.

Compactness in $\ell^2(\mathbb{Z}^n)$

For a bounded quaternionic left linear operator T on a Hilbert module its essential spectrum is defined as

$$\Sigma_{ess}(T) = \{s \in \mathbb{H} : T^2 - 2s_0 T + |s|^2 \text{ is Fredholm and its index is } 0\}.$$

Compactness

Let $\sigma \in S^0_{1,0}(\mathbb{Z}^n \times \mathbb{T}^n)$. Define $d := \limsup_{|x|\to\infty} \sup_{\xi\in\mathbb{T}^n} |\sigma(x,\xi)|$. Then $Op(\sigma)$ is compact on $\ell^2(\mathbb{Z}^n)$ if and only if $d = 0$.

Gohberg lemma

Let $\sigma \in S^0_{1,0}(\mathbb{Z}^n \times \mathbb{T}^n)$. Then for all compact operators K on $\ell_2(\mathbb{Z}^n)$ we have

$$\|Op(\sigma) - K\| \geq d.$$

Overview of the classical continuous case

Operator	symbol	class
Laplace operator Δ	$\sigma_\Delta(x,\xi) = -\lvert\xi\rvert^2$	$S^2_{1,0}$
Dirac operator D	$\sigma_D(x,\xi) = -i\xi$	$S^1_{1,0}$
Teodorescu operator T	$\sigma_T(x,\xi) = \frac{i\xi}{\lvert\xi\rvert^2}$	$S^{-1}_{1,0}$
Hilbert transform H	$\sigma_H(x,\xi) = \frac{i\xi}{\lvert\xi\rvert}$	$S^0_{1,0}$

. . . versus the discrete case

Operator	symbol $\sigma(x,\xi), \quad x \in h\mathbb{Z}^n, \xi \in \mathbb{T}^n$	class
star-Laplace operator Δ_h	$d^2 := \frac{4}{h^2} \sum_{j=1}^{n} \sin^2\left(\frac{h\xi_j}{2}\right)$	$S^0_{1,0}$
Dirac operator $D_h^{\mp}$	$\sum_{j=1}^{n} \left(\mathbf{e}_j^{+}\xi_{-j}^{D} + \mathbf{e}_j^{-}\xi_{+j}^{D}\right)$	$S^0_{1,0}$
Teodorescu operator T_h	$\frac{\sum_{j=1}^{n}\left(\mathbf{e}_j^{+}\xi_{-j}^{D}+\mathbf{e}_j^{-}\xi_{+j}^{D}\right)}{\frac{4}{h^2}\sum_{j=1}^{n}\sin^2\left(\frac{h\xi_j}{2}\right)}$	$S^0_{1,0}$
Hilbert transforms $H_{\pm}$	$\pm\frac{\widetilde{\xi}_{-}}{\underline{d}}\left(\mathbf{e}_n^{\mp}\frac{2}{h\underline{d}-\sqrt{4+h^2\underline{d}^2}} + \mathbf{e}_n^{\pm}\frac{h\underline{d}-\sqrt{4+h^2\underline{d}^2}}{2}\right)$	$S^0_{1,0}$

Possible applications?

In difference to the continuous case, the star-Laplacian and the discrete Dirac are now bounded operators.

Furthermore, none of the operators satisfies the compactness condition (in particular, the discrete Teodorescu operator).

Study of discrete Riemann-problems with non-constant coefficients.

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