Generalized Horn's double Hypergeometric Function and Associated Properties

Rakesh Kumar Parmar

Department of HEAS(Mathematics)

Bikaner Technical University, Bikaner-334004, India

Contact Information: Dr. Rakesh Kumar Parmar

Department of HEAS(Mathematics)

Bikaner Technical University

Bikaner-334004, Rajasthan, India

Phone: +919828506214

Email: rakeshparmar27@gmail.com

Abstract

Motivated by certain recent extensions of Euler's beta function, hypergeometric and confluent hypergeometric functions, we extend Horn's double hypergeometric function by making use of extended Euler's Beta function. We then systematic investigate to present its properties such as various integral representations of Euler and Laplace type, Mellin transforms, Laguerre polynomial representation, transformation formulae and a recurrence relation. Also, by means of Luke's bounds for hypergeometric functions and various bounds upon the Bessel functions appearing in the kernels of the newly established integral representations, we deduce a set of bounding inequalities for the extended Horn's double hypergeometric functions.

Introduction

In 1994, Chaudhry *et al.* [1, p. 20, Equation (1.7)] introduced the p–extended Beta function as:

$$B(x,\, y\,;\, p) := \int_0^1 t^{x-1}\,(1-t)^{y-1} \exp\left(-\frac{p}{t(1-t)}\right) dt \tag{1}$$

$$(\Re(p) > 0;\ \text{For } p = 0,\ \Re(x) > 0,\ \Re(y) > 0),$$

and studied various properties and obtained certain connections with Macdonald, error and Whittaker functions. Further more interesting properties and various connections with higher transcendental functions are investigated by Miller [6]. Here we call extended Beta function as p–extended Beta function.

In 2004, Chaudhry *et al.* [2] introduced the p–Gauss's hypergeometric and p–Kummer's confluent hypergeometric functions by making use of the p–extended Beta function $B(x,\, y\,;\, p)$ as:

$$F_p\,(a,\, b;\, c;\, z) := \sum_{n=0}^{\infty} (a)_n \frac{B(b+n,\, c-b;\, p)}{B(b,\, c-b)} \frac{z^n}{n!} \tag{2}$$

$$\big(p \geqq 0,\ |z| < 1\,;\ \Re(c) > \Re(b) > 0\big)$$

and

$$\Phi_p\,(b;\, c;\, z) := \sum_{n=0}^{\infty} \frac{B(b+n,\, c-b;\, p)}{B(b,\, c-b)} \frac{z^n}{n!} \tag{3}$$

$$\big(p \geqq 0\,;\ \Re(c) > \Re(b) > 0\big).$$

They investigated these functions with various properties including differentiation formulas, Mellin transform, transformation formulas, recurrence relations, summation formula, asymptotic formulas and certain interesting connections with some well known special functions..

Main Objectives

In terms of the extended beta function $B(x,\, y\,;\, p)$ defined by (1), we introduce extended Horn's Double hypergeometric function $H_{4,p}$ as follows: For $\alpha,\, \beta \in \mathbb{C}$ and $\gamma, \gamma' \in \mathbb{C} \setminus \mathbb{Z}_0^-$, we have

$$H_{4,p}[\alpha, \beta; \gamma, \gamma'; x, y] = \sum_{k,m=0}^{\infty} \frac{(\alpha)_{2k+m}}{(\gamma)_k} \frac{B(\beta+m, \gamma'-\beta; p)}{B(\beta, \gamma'-\beta)} \frac{x^k y^m}{k!\, m!} \tag{4}$$

$$\big(p \geqq 0;\ 2\sqrt{r} + s < 1,\ |x| \leq r,\ |y| \leq s \ \text{ when } \ p = 0\big).$$

Clearly, if put $p = 0$ in (4), we get the classical definition of Horn's Double hypergeometric function H_4 [5, 11]:

$$H_4[\alpha, \beta; \gamma, \gamma'; x, y] = \sum_{k,m=0}^{\infty} \frac{(\alpha)_{2k+m} (\beta)_m}{(\gamma)_k (\gamma')_m} \frac{x^k y^m}{k!\, m!}, \tag{5}$$

$$\big(2\sqrt{r} + s < 1,\ |x| \leq r,\ |y| \leq s\big)$$

. We investigate to present its properties such as various integral representations of Euler and Laplace type, Mellin transforms, Laguerre polynomial representation, transformation formulae and a recurrence relation. Also, by means of Luke's bounds for hypergeometric functions and various bounds upon the Bessel functions appearing in the kernels of the newly established integral representations, we deduce a set of bounding inequalities for the extended Horn's double hypergeometric functions.

Generalized Horn's Double hypergeometric function $H_{4,p}[x, y]$

Integral Representations

Theorem 1. The following Euler integral representation for $H_{4,p}$ in (4) holds true:

$$H_{4,p}[\alpha, \beta; \gamma; \gamma'; x, y] = \int_0^1 \frac{u^{\beta-1}(1-u)^{\gamma'-\beta-1}}{\mathrm{B}(\beta, \gamma'-\beta)\,(1-yu)^{\alpha}} \times {}_2F_1\left[\frac{\alpha}{2}, \frac{\alpha}{2}+\frac{1}{2}; \gamma';\ \frac{4x}{(1-yu)^2}\right] \exp\left(-\frac{p}{u(1-u)}\right) du \tag{6}$$

$$\big(\Re(p) > 0;\ \Re(\gamma') > \Re(\beta) > 0 \text{ when } p = 0\big).$$

Theorem 2. The following Laplace type integral representation for $H_{4,p}$ in (4) holds true:

$$H_{4,p}[\alpha, \beta; \gamma, \gamma'; x, y] = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{-t}\ {}_0F_1(-; \gamma; xt^2)\ \Phi_p(\beta; \gamma'; yt)\ dt \tag{7}$$

$$\big(\Re(p) > 0;\ \ \Re(\alpha) > 0 \text{ when } p = 0\big).$$

Mellin transform

Theorem 3. The following Mellin transform representation of $H_{4,p}$ in (4) holds true:

$$\mathcal{M}\left\{H_{4,p}[\alpha, \beta; \gamma, \gamma'; x, y] : p \to s\right\} = \frac{\Gamma(s) B(\beta+s, \gamma'+s-\beta)}{B(\beta, \gamma'-\beta)} \times H_4[\alpha, \beta+s; \gamma, \gamma'+2s; x, y] \tag{8}$$

$$(\Re(s) > 0 \text{ and } \Re(\beta+s) > 0,\ \Re(\gamma'+s-\beta) > 0)$$

.

Laguerre polynomial representation

Theorem 4. The following Laguerre polynomial representation holds true for $\Re(p) > 0$ and $\Re(\gamma'-\beta) > 0$:

$$H_{4,p}[\alpha, \beta; \gamma, \gamma'; x, y] = \frac{e^{-2p}}{B(\beta, \gamma'-\beta)} \sum_{m,n=0}^{\infty} B(\beta+m+1, \gamma'-\beta+n+1)\ H_{4,p}[\alpha, \beta+m+1; \gamma, \gamma'+m+n+2; x, y]\ L_m(p)\ L_n(p). \tag{9}$$

Bounding Inequalities

Theorem 5. Let $p > 0$ and for all $\Re(y) < 1$, $\Re(\alpha) > 0$ when $p = 0$. Then under $\alpha + 1 > \gamma > 0$ we have

$$\Big|H_{4,p}[\alpha, \beta; \gamma, \gamma'; -x, y]\Big| \leq \frac{\Gamma(\gamma)\,\Gamma(\alpha-\gamma+1)\,|x|^{\frac{1-\gamma}{2}}}{\sqrt{2}\,\Gamma(\alpha)\,\mathrm{e}^{4p}} \times \left\{1 - \frac{\beta}{\gamma'}\Big(1-(1-y)^{-\alpha+\gamma-1}\Big)\right\}. \tag{10}$$

In the same parameter range for all $x > 0$ there holds true

$$\Big|H_{4,p}[\alpha, \beta; \gamma, \gamma'; -x, y]\Big| \leq \frac{\Gamma(\gamma)\,\Gamma(\alpha-\gamma+1)\,b_L\,|x|^{\frac{1-\gamma}{2}}}{\sqrt[3]{\gamma-1}\,\Gamma(\alpha)\,\mathrm{e}^{4p}} \times \left\{1 - \frac{\beta}{\gamma'}\Big(1-(1-y)^{-\alpha+\gamma-1}\Big)\right\}. \tag{11}$$

Conclusions

In this present talk, we extend Horn's double hypergeometric function by making use of extended Euler's Beta function. We then systematic investigate to present its properties such as various integral representations of Euler and Laplace type, Mellin transforms, Laguerre polynomial representation, transformation formulae and a recurrence relation. Also, by means of Luke's bounds for hypergeometric functions and various bounds upon the Bessel functions appearing in the kernels of the newly established integral representations, we deduce a set of bounding inequalities for the extended Horn's double hypergeometric functions.

References

[1] M. A. Chaudhry, A. Qadir, M. Raflque, and S. M. Zubair, Extension of Euler's Beta function, *J. Comput. Appl. Math.* **78** (1997), 19–32.

[2] M. A. Chaudhry, A. Qadir, H. M. Srivastava and R. B. Paris, Extended hypergeometric and confluent hypergeometric functions, *Appl. Math. Comput.* **159** (2004), 589–602.

[3] M. A. Chaudhry and S. M. Zubair, *On a Class of Incomplete Gamma Functions with Applications*, CRC Press (Chapman and Hall), Boca Raton, FL, 2002.

[4] A. Erdélyi, W, Magnus, F. Oberhettinger and F. G. Tricomi, *Tables of Integral Transforms*, Vol. **I**, McGraw-Hill Book Company, New York, Toronto and London, 1954.

[5] J. Horn, Hypergeometrische Funktionen zweier Verändlichen, *Math. Ann.* **105** (1931), 381–407.

[6] Miller A. R., Remarks on generalized beta function, *J. Math. Anal. Appl.* **100** (1998), 23–32.

[7] Olver F. W. J., Lozier D. W., Boisvert R. F. and Clark C. W. (eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010.

[8] E. D. Rainville, *Special Functions*, Macmillan Company, New York, 1960; Reprinted by Chelsea Publishing Company, Bronx, New York, 1971.

[9] Slater L. J., *Confluent Hypergeometric Functions*, Cambridge University Press, Cambridge, London, and New York, 1960.

[10] H. M. Srivastava, Hypergeometric functions of three variables, *Ganita* **15** (1964), 97–108.

[11] H. M. Srivastava and Karlsson P. W., *Multiple Gaussian Hypergeometric Series*, Halsted Press, (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.

[12] Watson G. N., *A Treatise on the Theory of Bessel Functions*, Second edition, Cambridge University Press, Cambridge, London and New York, 1944.

[13] Whittaker E. T. and Watson G. N., *Modern Analysis*, Cambridge University Press, Cambridge, 1952.