Properties of Eigenfunctions of Pseudodifferential operators via Gabor frames

Elena Cordero

joint works with F. Bastianoni and F. Nicola (Politecnico di Torino)

Department of Mathematics
University of Torino

Advances in Pseudo-Differential Operators
Ghent Analysis & PDE Center
Ghent University
Ghent, July 7-8, 2020

Table of Contents

Goal of the study:

Given a compact pseudodifferential operator T on $L^2(\mathbb{R}^d)$ (or $L^2(G)$, G LCA group)

study conditions on its symbol which guarantee smoothness and decay of L^2-eigenfunctions

$$Tf = \lambda f, \quad \lambda \neq 0.$$

Localization operators

Localization operators have a longstanding tradition

- **Popular with the papers by I. Daubechies**
  [I. Daubechies Time-frequency localization operators: a geometric phase space approach, 1988. I. Daubechies, T. Paul, Time-frequency localization operators–a geometric phase space approach. II,1988.]
- **From then widely investigated by several authors in different fields of mathematics: from signal analysis to pseudodifferential calculus**
  [Abreu, Bayer, Cordoba, de Gosson, Dörfler, Fefferman, Gröchenig, Knutsen, Luef, Nicola, Pilipović, Ramanathan, Romero, Skrettingland, Teofanov, Toft, Topiwala, Wong...]
- **In quantum mechanics: already known as Anty-Wick operators**
  [Berezin, Wick and anti-Wick symbols of operators, Mat. Sb. 1971. Shubin, Pseudodifferential operators and spectral theory, 1980]

Applications: signal analysis, PDEs: approximations of pseudodifferential operators ("wave packets"), quantum mechanics: quantization procedure ("Anti-Wick operators")

Operators of translation and modulation

$$T_x f(t) = f(t-x), \quad M_\omega f(t) = e^{2\pi i \omega t} f(t), \; t, x, \omega \in \mathbb{R}^d.$$

Short-time Fourier transform (STFT)

$g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ fixed window function. The STFT of $f \in \mathcal{S}'(\mathbb{R}^d)$

$$V_g f(x,\omega) = \langle f, M_\omega T_x g \rangle = \int_{\mathbb{R}^d} f(t)\, \overline{g(t-x)}\, e^{-2\pi i \omega t}\, dt\,.$$

Fix $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$. The time-frequency (TF) localization operator $A_a^{\varphi_1,\varphi_2}$ with symbol $a \in \mathcal{S}'(\mathbb{R}^{2d})$ and windows φ_1, φ_2 is

$$A_a^{\varphi_1,\varphi_2} f(t) = \int_{\mathbb{R}^{2d}} a(x,\omega) V_{\varphi_1} f(x,\omega) M_\omega T_x \varphi_2(t)\, dx d\omega\,. \qquad (1)$$

If $a = \chi_\Omega$, $\Omega \subset \mathbb{R}^{2d}$ compact set, $\varphi_1 = \varphi_2$, then

$A_a^{\varphi_1,\varphi_2} f$**: the part of f that "lives on the set Ω"**

This is why $A_a^{\varphi_1,\varphi_2}$ called a TF localization operator.

Eigenfunctions of $A_a^{\varphi_1,\varphi_2}$

- Study of eigenvalues and eigenfunctions for $A_{\chi_\Omega}^{\varphi,\varphi}$, $\Omega \subset \mathbb{R}^{2d}$ compact domain, window $\varphi \in L^2(\mathbb{R}^d)$ by Abreu et al. [Dörfler, Gröchenig, Pereira, Romero, 2012, 2016, 2017], by Luef and Skrettingland [2018]:
  - Focus: asymptotic behaviour of the eigenvalues, depending on Ω
  - Tools: use self-adjoint operators:

$$(A_a^{\varphi_1,\varphi_2})^* = A_{\bar{a}}^{\varphi_2,\varphi_1}$$

    $\Rightarrow \varphi_1 = \varphi_2$ and the symbol a real valued
  - Symbol $a = \chi_\Omega$, Ω compact set
- Here the focus is the properties of eigenfunctions of $A_a^{\varphi_1,\varphi_2}$:
  - no requirement on the geometry of the symbol a (complex-valued, no compact support)
  - $A_a^{\varphi_1,\varphi_2}$ not necessarily self-adjoint (different windows φ_1, φ_2 allowed)

- Use modulation spaces as symbol classes
- Gabor frames to characterize eigenfunctions

Modulation spaces [Feichtinger 1983; Galperin-Samarah 2004]

Fix $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$, $0 < p, q \leq \infty$.

$$M^{p,q}(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{M^{p,q}} < \infty\},$$

$$\|f\|_{M^{p,q}} = \|V_g f\|_{L^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p dx\right)^{\frac{q}{p}} d\omega\right)^{\frac{1}{q}}$$

(obvious changes with $p = \infty$ or $q = \infty$).

- $\|f\|_{M^{p,q}}$ is a (quasi-)norm
- different windows $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ yield equivalent (quasi-)norms
- if $p = q$ write M^p instead of $M^{p,p}$,
- $L^p \hookrightarrow M^{p,\infty}$,
- $M^2 = L^2$.

- Inclusion relations [Feichtinger 1983, Galperin-Samarah 2004]:
  $0 < p_1 \leq p_2 \leq \infty$, $0 < q_1 \leq q_2 \leq \infty$.

$$M^{p_1,q_1}(\mathbb{R}^d) \hookrightarrow M^{p_2,q_2}(\mathbb{R}^d).$$

- Modulation spaces and Schawartz class:

$$\mathcal{S}(\mathbb{R}^d) \subsetneq \bigcap_{p>0} M^p(\mathbb{R}^d)$$

Theorem 1 (BCN 2019)

$a \in M^{p,\infty}(\mathbb{R}^{2d})$, $0 < p < \infty$, windows $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$.
$f \in L^2(\mathbb{R}^d)$:

$$A_a^{\varphi_1,\varphi_2} f = \lambda f, \lambda \neq 0$$

$\Rightarrow$

$$f \in \bigcap_{\gamma>0} M^\gamma(\mathbb{R}^d).$$

- Observe $\mathcal{S}(\mathbb{R}^d) \subsetneq \bigcap_{\gamma>0} M^\gamma(\mathbb{R}^d)$
- eigenfunctions extremely concentrated on the time-frequency space
- ...better explanation with Gabor frames

Representation of $A_a^{\varphi_1,\varphi_2}$ as Weyl operator

[C.-Gröchenig 2003, Shubin 1980]

$$\boxed{A_a^{\varphi_1,\varphi_2} = L_{a*W(\varphi_2,\varphi_1)}}^{1}$$

where $W(\varphi_2,\varphi_1)$ is the cross-Wigner distribution:

$$W(\varphi_2,\varphi_1)(x,\omega) = \int_{\mathbb{R}^d} \varphi_2(x+\frac{t}{2})\overline{\varphi_1(x-\frac{t}{2})}e^{-2\pi it\omega}\,dt.$$

Weyl symbol of $A_a^{\varphi_1,\varphi_2}$:

$$\boxed{\sigma = a * W(\varphi_2,\varphi_1).}$$

$\Rightarrow$

deduce properties for $A_a^{\varphi_1,\varphi_2}$ using Weyl form $L_{a*W(\varphi_2,\varphi_1)}$

1

$$L_\sigma f(x) = \int_{\mathbb{R}^{2d}} \sigma\left(\frac{x+y}{2},\omega\right) e^{2\pi i(x-y)\omega} f(y)\,dyd\omega$$

Properties of eigenfunctions of Weyl operators

Theorem (Toft 2004, Toft 2017)

$p, q, \gamma \in (0, \infty]$, $1/p + 1/q = 1/\gamma$
$\sigma \in M^{p,\min\{1,\gamma\}}(\mathbb{R}^{2d}) \Rightarrow L_\sigma : M^q(\mathbb{R}^d) \to M^\gamma(\mathbb{R}^d)$

Theorem (BCN 2019)

$f \in L^2(\mathbb{R}^d)$ is an eigenfunction of L_σ: $L_\sigma f = \lambda f, \quad \lambda \neq 0$.
Weyl symbol $\sigma \in M^{p,\gamma}$, some $0 < p < \infty$, $\forall \gamma > 0$

$$\Rightarrow$$

$$f \in \cap_{\gamma>0} M^\gamma(\mathbb{R}^d)$$

Sketch of proof.
$\sigma \in M^{p,\gamma}(\mathbb{R}^{2d})$, $\forall \gamma > 0 \Rightarrow L_\sigma : M^2(\mathbb{R}^d) \to M^{\gamma_1}(\mathbb{R}^d)$, $1/p + 1/2 = 1/\gamma_1$.
$p < \infty \Rightarrow \gamma_1 < 2$.
$f \in M^2(\mathbb{R}^d)$: eigenfunction with $\lambda \neq 0 \Rightarrow f = \frac{1}{\lambda} L_\sigma f \in M^{\gamma_1}(\mathbb{R}^d)$.
Starting now with $f \in M^{\gamma_1}(\mathbb{R}^d)$ and repeating the same argument $\Rightarrow$ eigenfunction $f \in M^{\gamma_2}(\mathbb{R}^d)$ (smaller), $1/p + 1/\gamma_1 = 1/\gamma_2$ ($\gamma_2 < \gamma_1$ since $p < \infty$). Continuing this way, construct decreasing sequence of indices $\gamma_n > 0$: $f \in M^{\gamma_n}(\mathbb{R}^d)$, $\lim_{n\to\infty} \gamma_n = 0$, $\Rightarrow f \in \cap_{\gamma>0} M^\gamma(\mathbb{R}^d)$.

The results of the previous theorem hold for any Shubin τ-representation $Op_\tau(\sigma)$, $\tau \in [0,1]$, defined by

$$Op_\tau(\sigma)f(x) = \int_{\mathbb{R}^{2d}} e^{2\pi i(x-y)\omega}\sigma((1-\tau)x+\tau y,\omega)f(y)\,dyd\omega.$$

For $\tau = 1/2 \Rightarrow Op_{1/2}(\sigma) = L_\sigma$ (we recapture the Weyl operator)
$a_1, a_2 \in \mathcal{S}'(\mathbb{R}^{2d})$, $\tau_1, \tau_2 \in [0,1]$, $\tau_1 \neq \tau_2$, [Hörmander, III, 1985]

$$Op_{\tau_1}(a_1) = Op_{\tau_2}(a_2) \Leftrightarrow a_2(x,\omega) = \frac{1}{|\tau_1-\tau_2|^d} e^{2\pi i(\tau_2-\tau_1)\Phi} * a_1(x,\omega),$$

where $\Phi(x,\omega) = x\omega$.
The mapping $a \mapsto T_\Phi a = e^{2\pi i\tau\Phi} * a$ is a homeomorphism on $M^{p,q}(\mathbb{R}^{2d})$, $1 \le p,q \le \infty$ [Toft, 2004]. The result easily extends to $0 < p,q \le \infty$ [Toft, 2017] (or use convolution relations for modulation spaces).

BCN 2019; C.-Gröchenig 2003, Toft 2004

$0 < p, q, r, t, u, \gamma \leq \infty$,

$$\frac{1}{u} + \frac{1}{t} = \frac{1}{\gamma}.$$

Assume

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}, \qquad \text{for } 1 \leq r \leq \infty$$

or

$$p = q = r, \qquad \text{for } 0 < r < 1.$$

$$\boxed{M^{p,u}(\mathbb{R}^d) * M^{q,t}(\mathbb{R}^d) \hookrightarrow M^{r,\gamma}(\mathbb{R}^d)}$$

norm inequality

$$\boxed{\|f * h\|_{M^{r,\gamma}} \lesssim \|f\|_{M^{p,u}} \|h\|_{M^{q,t}}}$$

Theorem 1. $a \in M^{p,\infty}(\mathbb{R}^{2d})$, $0 < p < \infty$, $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$.
$f \in L^2(\mathbb{R}^d)$: $A_a^{\varphi_1,\varphi_2} f = \lambda f, \lambda \neq 0 \Rightarrow$

$$f \in \bigcap_{\gamma > 0} M^{\gamma}(\mathbb{R}^d).$$

Ingredients for the proof:
(i) Properties of the Wigner distribution:
$\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d) \Rightarrow W(\varphi_2, \varphi_1) \in \mathcal{S}(\mathbb{R}^{2d}) \subset M^{r,\gamma}(\mathbb{R}^{2d})$, $0 < r, \gamma \leq \infty$.
(ii) Convolution relations for modulation spaces.
(iii) Properties of eigenfunctions of Weyl operators.

Gabor frames

$g \in L^2(\mathbb{R}^d) \setminus \{0\}$, $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$, $\alpha, \beta > 0$ (lattice of $\mathbb{R}^{2d}$). Define

- Gabor atoms [Gabor 1946]

$$g_{k,n} := M_{\beta n} T_{\alpha k} g, \quad k, n \in \mathbb{Z}^d.$$

  (time-frequency shifts of g)

- Gabor system $\mathcal{G}(g, \Lambda) = \{g_{k,n}, \quad k, n \in \mathbb{Z}^d\}$
- $\mathcal{G}(g, \Lambda)$ Gabor frame [Duffin-Schaeffer 1952]: $\exists\, A, B > 0$ such that

$$A\|f\|_2^2 \leq \sum_{k,n \in \mathbb{Z}^d} |\langle f, g_{k,n}\rangle|^2 \leq B\|f\|_2^2, \qquad \forall f \in L^2(\mathbb{R}^d).$$

if $\mathcal{G}(g, \Lambda)$ is a frame, the atoms $g_{k,n}$ are not orthogonal in general, but $\exists\ \gamma \in L^2(\mathbb{R}^d)$ (canonical dual window of g):

$$f = \sum_{k,n \in \mathbb{Z}^d} \langle f, g_{k,n}\rangle \gamma_{k,n} = \sum_{k,n \in \mathbb{Z}^d} \langle f, \gamma_{k,n}\rangle g_{k,n}.$$

A Gabor frame with canonical dual window $\gamma = g$ and $\|g\|_2 = 1$ is called *Parseval* frame (frame bounds $A = B = 1$):

$$f = \sum_{k,n\in\mathbb{Z}^d} \langle f, g_{k,n}\rangle g_{k,n}, \quad \|f\|_2^2 = \sum_{k,n\in\mathbb{Z}^d} |\langle f, g_{k,n}\rangle|^2.$$

Gröchenig 2001, Galperin-Samarah 2004

$0 < p, q \leq \infty$, $g, \gamma \in \mathcal{S}(\mathbb{R}^d)$ s.t. $\mathcal{G}(g, \Lambda)$ Gabor frame, γ dual window. Then $\exists\, 0 < A \leq B$:

$$A\|f\|_{M^{p,q}} \leq \left(\sum_{n\in\mathbb{Z}^d} \left(\sum_{k\in\mathbb{Z}^d} |\langle f, g_{k,n}\rangle|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \leq B\|f\|_{M^{p,q}},$$

$\forall f \in M^{p,q}(\mathbb{R}^d)$. That is,

$$\|f\|_{M^{p,q}} \asymp \|(\langle f, g_{k,n}\rangle)_{k,n}\|_{\ell^{p,q}}.$$

High compression of eigenfunctions onto Gabor atoms

$$f \in \bigcap_{\gamma>0} M^{\gamma}(\mathbb{R}^d)$$

Eigenfunctions extremely concentrated on the time-frequency space: very few Gabor coefficients large, all the others negligible

Try to explain it better ...

$g \in \mathcal{S}(\mathbb{R}^d)$: $\{g_{k,n}\}_{k,n}$ Parseval Gabor frame

$$\Sigma_N = \left\{ p = \sum_{k,n \in F} c_{k,n} g_{k,n} \ : \ c_{k,n} \in \mathbb{C}, \ F \subset \mathbb{Z}^d \times \mathbb{Z}^d, \ \text{card}\, F \leq N \right\}$$

(set of all linear combinations of Gabor atoms with at most N terms).

$N \in \mathbb{N}_+$, $f \in L^2(\mathbb{R}^d)$, N-term approximation error $\sigma_N(f)$:

$$\sigma_N(f) = \inf_{p \in \Sigma_N} \|f - p\|_2.$$

$\sigma_N(f)$ error produced when f is approximated optimally by a linear combination of N Gabor atoms

$\Rightarrow$

Proposition

$f \in M^p(\mathbb{R}^d)$ some $0 < p < 2$. $\exists\, C = C(p) > 0$:

$$\sigma_N(f) \le C\|f\|_{M^p(\mathbb{R}^d)} N^{-\gamma},$$

where $\gamma = 1/p - 1/2 > 0$.

Corollary

Any f eigenfunction of $A_a^{\varphi_1,\varphi_2}$ (eigenvalue $\lambda \neq 0$) is highly compressed onto a few Gabor atoms $g_{k,n}$: $\forall r > 0\ \exists\, C = C(r) > 0$:

$$\sigma_N(f) \le C N^{-r}.$$

N-term approximation error $\sigma_N(f)$ presents super-polynomial decay

Pioneering papers:

- Feichtinger and Kozek. Quantization of TF-lattice invariant operators on elementary LCA groups.
- Gröchenig. Aspects of Gabor analysis on locally compact abelian groups. [In: Gabor analysis and algorithms. Appl. Numer. Harmon. Anal. Birkhäuser Boston, Boston, MA, 1998]
- Gröchenig and Strohmer. Pseudodifferential operators on locally compact abelian groups and Sjöstrand's symbol class. [Journal für die reine und angewandte Mathematik, 2007]

Main motivation: Applications use discrete signals and numerical implementations require to consider finite periodic signals and consequently Gabor theory on finite cyclic groups (p-adic groups)

Many authors have been working on LCA groups (Enstad, Feichtinger, Havin, He, Jakobsen, Kaliszewski, Kaniuth, King, Kutyniok, Luef, Nikolskij, Omland, Quigg, Skopina, Wong, ...)

$\mathcal{G}$ LCA group, countable union of compact sets and metrizable ($\Leftrightarrow L^2(\mathcal{G})$ is separable) $\widehat{\mathcal{G}}$ dual group of $\mathcal{G}$, the action of a character $\omega \in \widehat{\mathcal{G}}$ on an element $x \in \mathcal{G}$ is denoted by $\langle \omega, x \rangle$.
$\sigma \in M^\infty(\mathcal{G} \times \widehat{\mathcal{G}})$, the pseudodifferential operator with Kohn-Nirenberg symbol σ is the operator

$$K_\sigma f(x) = \int_{\widehat{\mathcal{G}}} \sigma(x, \omega) \hat{f}(\omega)^{2} \langle \omega, x \rangle d\omega$$

The (cross-) Rihaczek distribution of $f, g \in L^2(\mathcal{G})$ is defined as

$$R(f, g)(x, \omega) = f(x) \overline{\hat{g}(\omega) \langle \omega, x \rangle}$$

and

$$\langle K_\sigma f, g \rangle = \langle \sigma, R(g, f) \rangle.$$

[2] $\hat{f}(\omega) = \int_{\widehat{G}} f(x) \overline{\langle \omega, x \rangle} dx$

Theorem

$p_i, q_i, p, q \in (0,\infty]$, $i = 1,2$,

$$\min\{\frac{1}{p_1} + \frac{1}{p_2'}, \frac{1}{q_1} + \frac{1}{q_2'}\} \geq \frac{1}{p'} + \frac{1}{q'}. \tag{2}$$

and we have

$$q \leq \min\{p_1', q_1', p_2, q_2\}. \tag{3}$$

If $\sigma \in M^{p,q}(\mathcal{G} \times \widehat{\mathcal{G}})$ then K_σ is a bounded operator from $M^{p_1,q_1}(\mathcal{G})$ to $M^{p_2,q_2}(\mathcal{G})$, with the estimate

$$\|K_\sigma f\|_{M^{p_2,q_2}} \lesssim \|\sigma\|_{M^{p,q}} \|f\|_{M^{p_1,q_1}}. \tag{4}$$

The result is sharp, as shown for $\mathcal{G} = \mathbb{R}^d$ [C-Nicola, 2018 (Banach case), C., 2020 (quasi-Banach case)]

Further works:

- Gabor almost diagonalization of K_σ with symbols $\sigma \in M^{\infty,p}$, $0 < p \leq 1$
- Wiener (quasi-)algebras properties for K_σ with $\sigma \in M^{\infty,p}$, $0 < p \leq 1$
(generalization of the Sjöstrand's class $M^{\infty,1}$ [Gröchenig, 2006, Gröchenig-Strohmer, 2008])
- Properties of eigenfunctions for K_σ and localization operators
(conjecture: similar results to the case $\mathcal{G} = \mathbb{R}^d$)

Thank you for your attention!