Introduction into fractional calculus
Beginnings of fractional differentiation: prehistory

- The idea to generalize notion \( \frac{d^p f(x)}{dx^p} \) to non-integer values of \( p \) appeared at the birth of differential calculus itself in Leibniz correspondence with Bernoulli who asked about meaning of the theorem on differentiation of product of functions for non-integer value of differentiation.

- Leibniz in his letters to L’Hopital (1695) and to Wallis (1697) made remarks of possibility to consider differentiation of order \( 1/2 \).

- Euler (1738) was the one who took first step by observation that there is a sense to consider \( \frac{d^p x^a}{dx^p} \) for non-integer \( p \).

- Laplace (1812) proposed the differentiation of functions representable in the form \( \int T(t)t^{-x}dt \).

- Lacroix (1820) repeated idea of Euler and gave the formula for \( \frac{d^{1/2} x^a}{dx^{1/2}} \).
Beginings of fractional differentiation

- Fourier (1822) give the idea for the formula to define non-integer differentiation:

\[
\frac{d^p f(x)}{dx^p} = \frac{1}{2\pi} \int \lambda^p d\lambda \int f(t) \cos(\lambda(x - t) + p\frac{\pi}{2}) dt
\]

- Proper history starts with Abel and Liouville ...

- Brief historical outline is given in the comprehensive book on fractional integrals and derivatives:

Figure 2 shows the histograms for the books with author and books edited indices. The exponential trendlines reveal a good correlation factor, but different growing rates. This discrepancy means that such indices describe only a part of the object under analysis and common sense suggests that some value in between both cases is probably closer to the “true”. These trend lines reflect the past and there is no guarantee that we can foresee the future as the Moore law seems to be demonstrating recently.
The Riemann-Liouville fractional calculus

Let $u \in L^1([a, b])$ and $\alpha > 0$.

- The left Riemann-Liouville fractional integral of order $\alpha$:
  \[
  a l_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \theta)^{\alpha - 1} u(\theta) \, d\theta.
  \]

- The right Riemann-Liouville fractional integral of order $\alpha$:
  \[
  t l_b^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\theta - t)^{\alpha - 1} u(\theta) \, d\theta.
  \]

(The gamma function: $\Gamma(z) := \int_0^\infty e^{-t} t^{z-1} \, dt$, $\Re z > 0$)

- Semigroup property: $a l_t^\alpha a l_t^\beta u = a l_t^{\alpha + \beta} u$ and $t l_b^\alpha t l_b^\beta u = t l_b^{\alpha + \beta} u$

Consequences:

- $a l_t^\alpha$ and $a l_t^\beta$ commute, i.e., $a l_t^\alpha a l_t^\beta = a l_t^\beta a l_t^\alpha$;

- $a l_t^\alpha (t - a)^\mu = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + 1 + \alpha)} \cdot (t - a)^{\mu + \alpha}$. 

\[1/17\]
Let \( u \in AC([a, b]) \) and \( 0 \leq \alpha < 1 \).

- The left Riemann-Liouville fractional derivative of order \( \alpha \):
  \[
  aD_t^\alpha u(t) = \frac{d}{dt} \left( aI_t^{1-\alpha} u(t) \right) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{u(\theta)}{(t-\theta)^\alpha} d\theta.
  \]

- The right Riemann-Liouville fractional derivative of order \( \alpha \):
  \[
  tD_b^\alpha u(t) = \left( -\frac{d}{dt} \right) tI_b^{1-\alpha} u(t) = \frac{1}{\Gamma(1-\alpha)} \left( -\frac{d}{dt} \right) \int_t^b \frac{u(\theta)}{(\theta-t)^\alpha} d\theta.
  \]

- If \( \alpha = 0 \) then \( aD_t^0 u(t) = tD_b^0 u(t) = u(t) \).
- When \( \alpha \to 1^- \), \( aD_t^\alpha u(t) \to u'(t) \) and \( tD_b^\alpha u(t) \to -u'(t) \).
- Euler formula:
  \[
  aD_t^\alpha (t-a)^{-\mu} = \frac{\Gamma(1-\mu)}{\Gamma(1-\mu-\alpha)} \frac{1}{(t-a)^{\mu+\alpha}}
  \]

Consequences:
- \( \mu = 1-\alpha \): \( aD_t^\alpha (t-a)^{\alpha-1} = 0 \)
- \( \mu = 0 \): \( aD_t^\alpha c \neq 0 \), for any constant \( c \in \mathbb{R} \).
Definition of R-L fractional derivatives can be extended for any $\alpha \geq 1$: write $\alpha = [\alpha] + \{\alpha\}$, where $[\alpha]$ denotes the integer part and $\{\alpha\}$, $0 \leq \{\alpha\} < 1$, the fractional part of $\alpha$. Let $u \in AC^n([a, b])$ and $n - 1 \leq \alpha < n$.

- The left Riemann-Liouville fractional derivative of order $\alpha$:

$$aD_t^\alpha u(t) = \left(\frac{d}{dt}\right)^{[\alpha]} aD_t^\{\alpha\} u(t) = \left(\frac{d}{dt}\right)^{[\alpha]+1} aI_t^{1-\{\alpha\}} u(t)$$

or

$$aD_t^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{u(\theta)}{(t-\theta)^{\alpha-n+1}} d\theta$$

- The right Riemann-Liouville fractional derivative of order $\alpha$:

$$tD_b^\alpha u(t) = \left(-\frac{d}{dt}\right)^{[\alpha]} tD_b^\{\alpha\} u(t) = \left(-\frac{d}{dt}\right)^{[\alpha]+1} tI_b^{1-\{\alpha\}} u(t)$$

or

$$tD_b^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt}\right)^n \int_t^b \frac{u(\theta)}{(\theta-t)^{\alpha-n+1}} d\theta$$
Properties:

- \( a D_t^\alpha a l_t^\alpha u = u \)
- \( a l_t^\alpha a D_t^\alpha u \neq u \), but

\[ a l_t^\alpha a D_t^\alpha u(t) = u(t) - \sum_{k=0}^{n-1} \frac{(t - a)^{\alpha - k - 1}}{\Gamma(\alpha - k)} \frac{d^{n-k-1}}{dt} \bigg|_{t=a} \left( a l_t^{n-\alpha} u(t) \right) \]

Contrary to fractional integration, Riemann-Liouville fractional derivatives do not obey either the semigroup property or the commutative law: E.g. \( u(t) = t^{1/2}, a = 0, \alpha = 1/2 \) and \( \beta = 3/2 \).

\[ 0 D_t^\alpha u = \Gamma\left(\frac{3}{2}\right) = \frac{1}{2} \sqrt{\pi}, \quad 0 D_t^\beta u = 0 \]

\[ 0 D_t^\alpha\left(0 D_t^\beta u\right) = 0, \quad 0 D_t^\beta\left(0 D_t^\alpha u\right) = -\frac{1}{4} t^{-\frac{3}{2}}, \quad 0 D_t^{\alpha+\beta} u = -\frac{1}{4} t^{-\frac{3}{2}} \]

\[ \Rightarrow 0 D_t^\alpha\left(0 D_t^\beta u\right) \neq 0 D_t^{\alpha+\beta} u \quad \text{and} \quad 0 D_t^\alpha\left(0 D_t^\beta u\right) \neq 0 D_t^\beta\left(0 D_t^\alpha u\right) \]
Left Riemann-Liouville fractional operators via convolutions

In the distributional setting: \((\mathcal{S}_+ - \text{space of Schwartz' distributions supported in } [0, \infty))\)

\[ f_\alpha(t) := \begin{cases} 
H(t) \frac{t^{\alpha-1}}{\Gamma(\alpha)}, & \alpha > 0 \\
\left(\frac{d}{dt}\right)^N f_{\alpha+N}(t), & \alpha \leq 0, N \in \mathbb{N} : N + \alpha > 0 
\end{cases} \quad (\in \mathcal{S}_+) \]

Then:

- \(\alpha \in \mathbb{N} : f_1(t) = H(t), \ f_2(t) = t_+, \ldots \ f_n(t) = \frac{t_+^{n-1}}{(n-1)!}\)
- \(-\alpha \in \mathbb{N}_0 : f_0(t) = \delta(t), \ f_{-1}(t) = \delta'(t), \ldots \ f_{-n}(t) = \delta^{(n)}(t)\)

The convolution operator \(f_\alpha \ast\) is the left Riemann-Liouville operator of
- differentiation for \(\alpha < 0\)
- integration for \(\alpha \geq 0\)

For \(0 < \alpha < 1\):
- \(0D_t^\alpha u(t) = f_{-\alpha} \ast u(t)\)
- \(\mathcal{L}[0D_t^\alpha u(t)](s) = s^\alpha \mathcal{L}u(s) = s^\alpha \tilde{u}(s) \quad (s \in \mathbb{C})\)
- \(\mathcal{F}[0D_t^\alpha u(t)](\omega) = (i\omega)^\alpha \mathcal{F}u(\omega) = (i\omega)^\alpha \tilde{u}(\omega) \quad (\omega \in \mathbb{R})\)
Fractional identities

- **Linearity:**
  \[ a D_t^\alpha (\mu u + \nu v) = \mu \cdot a D_t^\alpha u + \nu \cdot a D_t^\alpha v \]

- The Leibnitz rule does not hold in general:
  \[ a D_t^\alpha (u \cdot v) \neq u \cdot a D_t^\alpha v + a D_t^\alpha u \cdot v \]

- For analytic functions \( u \) and \( v \):
  \[ a D_t^\alpha (u \cdot v) = \sum_{i=0}^{\infty} \binom{\alpha}{i} (a D_t^{\alpha-i} u) \cdot v^{(i)}, \quad \binom{\alpha}{i} = \frac{(-1)^{i-1} \alpha \Gamma(i - \alpha)}{\Gamma(1 - \alpha) \Gamma(i + 1)} \]

- For a real analytic function \( u \):
  \[ a D_t^\alpha u = \sum_{i=0}^{\infty} \binom{\alpha}{i} \frac{(t - a)^{i-\alpha}}{\Gamma(i + 1 - \alpha)} u^{(i)}(t) \]
Integration by parts formulae

Let $0 < \alpha < 1$.

- For $f \in L^p(a, b)$ and $g \in L^q(a, b)$, $p \geq 1$, $q \geq 1$, $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$, we have
  \[
  \int_a^b a I_t^\alpha f(x) g(x) \, dx = \int_a^b f(x) t I_b^\alpha g(x) \, dx.
  \]

- For $f \in t I_b(L^p(a, b))$ and $g \in L^q(a, b)$, $p \geq 1$, $q \geq 1$, $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$, we have
  \[
  \int_a^b a D_t^\alpha f(x) g(x) \, dx = \int_a^b f(x) t D_b^\alpha g(x) \, dx.
  \]
Caputo fractional derivatives

Let $u \in AC^n([a, b])$ and $n - 1 \leq \alpha < n$.

- The left Caputo fractional derivative of order $\alpha$:

$$c_a D_t^\alpha u(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t \frac{u^{(n)}(\theta)}{(t - \theta)^{\alpha-n+1}} d\theta$$

- The right Caputo fractional derivative of order $\alpha$:

$$c_t D_b^\alpha u = \frac{1}{\Gamma(n - \alpha)} \int_t^b \frac{(-u)^{(n)}(\theta)}{(\theta - t)^{\alpha-n+1}} d\theta$$

- In general, R-L and Caputo fractional derivatives do not coincide:

$$a D_t^\alpha u = c_a D_t^\alpha u + \sum_{k=0}^{n-1} \frac{(t-a)^{k-\alpha}}{\Gamma(k - \alpha + 1)} u^{(k)}(a^+).$$

If $u^{(k)}(a^+) = 0$, $(k = 0, 1, \ldots, n - 1)$ or $u \in S'_+$ then $a D_t^\alpha u = c_a D_t^\alpha u$. 
Symmetrized fractional derivatives: generalisation of the first order

Let \( u \in AC([a, b]) \), and \( 0 < \alpha < 1 \).

- \( \frac{c}{a} \mathcal{E}_b^\beta u(x) = \frac{1}{2}(\frac{c}{a} D_x^\beta u(x) - \frac{c}{x} D_b^\beta u(x)) = \frac{1}{\Gamma(1 - \beta)} \int_a^b \frac{u'(\theta)}{|t - \theta|^\beta} d\theta \).
- For \( a = -\infty \) and \( b = \infty \)
  \[
  \mathcal{E}_x^\beta u(x) = \frac{1}{2} \frac{1}{\Gamma(1 - \beta)} |x|^{-\beta} \ast u'(x).
  \]
- \( \mathcal{E}_x^0 u(x) = 0 \) and \( \mathcal{E}_x^\beta u(x) \rightarrow u'(x) \), as \( \beta \rightarrow 1 \).
- \( \mathcal{F} \left( \mathcal{E}_x^\beta u(x) \right) = i \sin\left(\frac{\beta \pi}{2}\right) \text{sgn}(\xi) |\xi|^\beta \mathcal{F} u(\xi) \)
Symmetrized fractional derivatives: generalisation of the second order derivative

- of order $\alpha \in (1, 2)$

$$\mathcal{E}_x^\alpha u = \frac{1}{2} (D_+^\alpha + D_-^\alpha) u = \frac{1}{2\Gamma (2 - \alpha) |x|^{\alpha - 1}} * \frac{d^2}{dx^2} u(x).$$

- of order $\alpha \in (2, 3)$

$$\mathcal{E}_x^\alpha u = \frac{1}{2} (D_+^\alpha + D_-^\alpha) u = \frac{1}{2\Gamma (3 - \alpha) |x|^{\alpha - 2}} * \frac{d^3}{dx^3} u(x).$$

- In both cases

$$\mathcal{F} (\mathcal{E}_x^\alpha w) = |\xi|^{\alpha} \cos(\frac{\alpha \pi}{2}) \hat{w}$$

so for the definition we can take

$$\mathcal{E}_x^\alpha w = \mathcal{F}^{-1} (|\xi|^{\alpha} \cos(\frac{\alpha \pi}{2})).$$
Distributed order fractional derivative

For $\phi \in \mathcal{E}'(\mathbb{R})$ and $\varphi \in \mathcal{S}(\mathbb{R})$ one defines distributed order fractional derivative $\int_{\text{supp} \phi} \phi(\alpha) D^\alpha u(t) \, d\alpha$ as follows:

$$\langle \int_{\text{supp} \phi} \phi(\alpha) D^\alpha u(t) \, d\alpha, \varphi(t) \rangle := \langle \phi(\alpha), \langle D^\alpha u(t), \varphi(t) \rangle \rangle,$$

- $\gamma \rightarrow \langle D^\gamma u(t), \varphi(t) \rangle : \mathbb{R} \rightarrow \mathbb{R}$ is smooth.

- $\mathcal{L} \left( \int_{\text{supp} \phi} \phi(\alpha) D^\alpha u(t) \, d\alpha \right)(s) = \tilde{u}(s) \langle \phi(\alpha), s^\alpha \rangle$ (Re $s > 0$)
Books: Theory...

Oldham, K.B., Spanier, J.
The Fractional Calculus. Academic Press

S. G. Samko, A. A. Kilbas, and O. I. Marichev.
Fractional Integrals and Derivatives.

Podlubny, I.
Fractional Differential Equations,

K. Diethelm.
The Analysis of Fractional Differential Equations.
Books: ... and Applications

Miller, K.S., Ross, B.
An Introduction to the Fractional Integrals and Derivatives - Theory and Applications

Gorenflo, R., Mainardi, F.
Fractional calculus: Integral and differential equations of fractional order

Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.
Theory and Applications of Fractional Differential Equations

T. M. Atanackovicc, S. Pilipovicc, B. Stankovicc, D.Zorica
Fractional Calculus with Applications in Mechanics. Vol 1 and Vol 2
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