Fractional-order operators on nonsmooth domains

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Plan:

1. Classical theory.
2. Fractional-order operators.
3. Nonsmooth coordinate changes.
4. Application to fractional-order boundary problems.

A pseudodifferential operator (ψdo) $P = \mathrm{Op}(p)$ is defined on $\mathbb{R}^n$ by:

$$\mathrm{Op}(p(x,\xi))u(x) = \tfrac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi)\, d\xi,$$

using the Fourier transformation $\mathcal{F}u = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\, dx$.

Classical symbols: $S^m(\mathbb{R}^n \times \mathbb{R}^n)$ consists of C^∞-functions $p(x,\xi)$ such that (with $\langle\xi\rangle = (|\xi|^2+1)^{\frac12}$),

- $p \sim \sum_{j\in\mathbb{N}_0} p_j$, $p_j(x,t\xi) = t^{m-j} p_j(x,\xi)$ for $|\xi| \ge 1$, $j \in \mathbb{N}_0$, and

$$\partial_x^\beta \partial_\xi^\alpha \big(p(x,\xi) - \sum\nolimits_{j<J} p_j(x,\xi)\big) \text{ is } O(\langle\xi\rangle^{m-J-|\alpha|}), \text{ all } J, \alpha, \beta.$$

When $\tau > 0$, C^τ denotes the Hölder space $C^{k,\sigma}$ if $\tau = k + \sigma$ with $k \in \mathbb{N}_0$ and $\sigma \in (0,1)$, and the space C_b^k if $\tau = k \in \mathbb{N}$.

Classical C^τ-symbols: $C^\tau S^m(\mathbb{R}^n \times \mathbb{R}^n)$ consists of functions $p(x,\xi)$ that are C^τ in x and C^∞ in ξ (same for $\partial_\xi^\alpha p$) such that

- $p \sim \sum_{j\in\mathbb{N}_0} p_j$, $p_j(x,t\xi) = t^{m-j} p_j(x,\xi)$ for $|\xi| \geq 1$, $j \in \mathbb{N}_0$, and

$$\|\partial_\xi^\alpha \big(p(\cdot,\xi) - \sum\nolimits_{j<J} p_j(\cdot,\xi)\big)\|_{C^\tau(\mathbb{R}^n)} \text{ is } O(\langle\xi\rangle^{m-J-|\alpha|}), \text{ all } J, \alpha.$$

(Initiated by Kumanogo-Nagase '78, Marschall '87, ...)

L_p Sobolev-type spaces $H_p^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \mathcal{F}^{-1}(\langle\xi\rangle^s \hat{u}) \in L_p(\mathbb{R}^n)\}$.

Mapping property: When $p \in C^\tau S^m(\mathbb{R}^n \times \mathbb{R}^n)$,

$$P\colon H_p^{s+m}(\mathbb{R}^n) \to H_p^s(\mathbb{R}^n), \text{ for } |s| < \tau,\ 1 < p < \infty.$$

Boundary value theories: $\Omega \subset \mathbb{R}^n$ smooth bounded, or $= \mathbb{R}^n_+$. A calculus initiated by Boutet de Monvel '71, Rempel-Schulze '82, G. '84,.... Applies to **integer-order** operators satisfying the 0-transmission condition at $\partial\Omega$:

$$\partial_x^\beta \partial_\xi^\alpha p_j(x, -\nu(x)) = e^{i\pi(m-j-|\alpha|)} \partial_x^\beta \partial_\xi^\alpha p_j(x, \nu(x)), \qquad \text{(0-tr)}$$

for $x \in \partial\Omega$, $\nu(x)$ the interior normal at x, all indices.

This includes differential operators and their solution operators.
Let r^+ denote restriction from $\mathbb{R}^n$ to $\mathbb{R}^n_+$ (or from $\mathbb{R}^n$ to Ω), e^+ extension by zero from $\mathbb{R}^n_+$ to $\mathbb{R}^n$ (or from Ω to $\mathbb{R}^n$), and define $P_+ = r^+Pe^+$, acting on $\mathbb{R}^n_+$ (resp. Ω). When P of order $m \in \mathbb{Z}$ satisfies (0-tr), then

$$P_+ = r^+Pe^+ \colon \overline{H}_p^{s+m}(\Omega) \to \overline{H}_p^s(\Omega) \text{ for } s > -m - \tfrac{1}{p'}.$$

We here refer to

$$\overline{H}_p^s(\Omega) = r^+H_p^s(\mathbb{R}^n), \text{ the } \textit{restricted} \text{ space},$$
$$\dot{H}_p^s(\overline{\Omega}) = \{u \in H_p^s(\mathbb{R}^n) \mid \operatorname{supp} u \subset \overline{\Omega}\}, \text{ the } \textit{supported} \text{ space}.$$

The Boutet de Monvel calculus also contains other operators, such as trace operators (from Ω to $\partial\Omega$) and Poisson operators (from $\partial\Omega$ to Ω) and their interplay.
Helmut Abels '05 extended the calculus to $C^\tau S^m(\mathbb{R}^n \times \mathbb{R}^n)$ symbols, showing that here

$$P_+ = r^+Pe^+ \colon \overline{H}_p^{s+m}(\Omega) \to \overline{H}_p^s(\Omega) \text{ for } s > -m - \tfrac{1}{p'}, |s| < \tau.$$

Applications to Navier-Stokes problems by Abels and coauthors.

2. Fractional-order operators

An interesting example that has been the subject of much research recently is the fractional Laplacian $(-\Delta)^a$, $0 < a < 1$. It is a ψdo $\mathrm{Op}(|\xi|^{2a})$, but some people prefer to see it as a singular integral operator:

$$(-\Delta)^a u(x) = c_{n,a} PV \int_{\mathbb{R}^n} \frac{u(x) - u(x+y)}{|y|^{n+2a}}\, dy;$$

here $c_{n,a}|y|^{-n-2a} = \mathcal{F}^{-1}|\xi|^{2a}$. Of great interest in probability theory and finance (and mathematical physics and differential geometry).

Interesting x-dependent ψdo generalizations are $P = L^a$, L a second-order strongly elliptic differential operator $-\sum a_{jk}(x)\partial_j\partial_k + \sum b_k(x)\partial_k + c(x)$. The Boutet de Monvel calculus does not apply. But one can define a *homogeneous Dirichlet problem*

$$r^+ Pu = f \text{ in } \Omega, \quad \operatorname{supp} u \subset \overline{\Omega},$$

where for example f is given in $L_2(\Omega)$ and u is sought in $\dot{H}_2^a(\overline{\Omega})$. When P is strongly elliptic, one can define a Fredholm operator P_{Dir} representing this problem in $L_2(\Omega)$, with domain

$$D(P_{\mathrm{Dir}}) = \{u \in \dot{H}_2^a(\overline{\Omega}) \mid r^+ Pu \in L_2(\Omega)\}.$$

The **regularity question**: Is u more regular? When f is more regular?

Ros-Oton and Serra showed in 2014 by potential-theoretic methods:

$$f \in L_\infty(\Omega) \implies u \in d^a \overline{C}^t(\Omega), \text{ for small } t > 0,$$

where $d(x) = \operatorname{dist}(x, \partial\Omega)$ near $\partial\Omega$ (extended > 0 to Ω), Ω being $C^{1,1}$.
G. 2014/15 showed, when Ω is C^∞:

$$f \in \overline{H}_p^s(\Omega) \implies u \in \dot{H}_p^{s+2a}(\overline{\Omega}) + d^a \overline{H}_p^{s+a}(\Omega), \text{ when } s > -a - \tfrac{1}{p'},$$
$$f \in C^\infty(\overline{\Omega}) \iff u \in d^a C^\infty(\overline{\Omega}).$$

This was based on a new theory developed from an informal lecture note of Hörmander 1966, for ψdo's satisfying the a-transmission condition:

$$\partial_x^\beta \partial_\xi^\alpha p_j(x, -\nu(x)) = e^{i\pi(m-2a-j-|\alpha|)} \partial_x^\beta \partial_\xi^\alpha p_j(x, \nu(x)), \qquad (a\text{-tr})$$

at $\partial\Omega$. Many more consequences have been developed, e.g. studies that shows how the traces and normal derivatives of u/d^a and u/d^{a-1} have a role as nonhomogeneous boundary values.
Similar properties were shown in Hölder spaces.

Both the C^∞-result and the results for H_p^s-spaces with higher s (and $p \neq 2$) were new, and show the power of the ψdo methods.

3. Nonsmooth coordinate changes

But, when I have lectured on these things for people in applications, I have been met by the question: What can the ψdo methods do when Ω is nonsmooth?

Some results can be obtained by approximation. But the tool of **localization**, change-of-variables — which would carry the operator at a curved boundary over to an operator at $\mathbb{R}^n_+$ with full symbol — has been missing in nonsmooth cases. I have been working with Helmut Abels on this for several years, completed recently.

The existing theories for operators with $C^\tau S^m(\mathbb{R}^n \times \mathbb{R}^n)$-symbols have composition rules and parametrix constructions, but handle only smooth coordinate changes. E.g. Marschall '88 has the formula for the symbol after a coordinate change with $[\tau]$ precise terms, but a remainder that is only in a usable form when the coordinate change is smooth.

Recall the formulas in the smooth case: For a diffeomorphism $F\colon \mathbb{R}^n \to \mathbb{R}^n$, denote $(v \circ F)(y) = (F^* v)(y)$ and $(u \circ F^{-1})(x) = (F^{*,-1} u)(x)$. A ψdo $P = \operatorname{Op}(p)$ on $\mathbb{R}^n$ gives rise to an operator $\underline{P}$ by

$$(\underline{P} u)(x) = (P(u \circ F^{-1}))(F(x)) = (F^* P F^{*,-1} u)(x).$$

For simplicity, let $\sup_x |\nabla F(x) - I| \le \frac{1}{2}$. Define

$$A(x,y) = \int_{[0,1]} \nabla_x F(x + t(y-x))\, dt,$$
$$q(x,y,\xi) = p(F(x), A(x,y)^{-1,T}\xi)|\det A(x,y)|^{-1}|\det \nabla_y F(y)|,$$

for $x, y, \xi \in \mathbb{R}^n$. Then

$$\underline{P}u = \mathrm{Op}(q(x,y,\xi))u = \tfrac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} q(x,y,\xi)u(y)\, dyd\xi. \quad (\star)$$

This operator "in (x,y)-form" is reduced to x-form $\underline{P} = \mathrm{Op}(\underline{p}(x,\xi))$ by

$$\underline{p}(x,\xi) = \sum\nolimits_{|\alpha|<N} \tfrac{1}{\alpha!}\partial_y^\alpha D_\xi^\alpha q(x,y,\xi)\big|_{y=x} + r_N(x,\xi). \quad (\star\star)$$

When F is C^∞, this extends to $p \in C^\tau S^m$ for $N < \tau$, with $r_N \in C^{\tau-N} S^{m-N}$.

For the case where F is merely $C^{\tau+1}$, $\tau > 0$, it has required a large effort in Abels-G. '20 to extend the calculations. The crucial step was to validate $(\star)$ as an oscillatory integral, and to show a reduction from (x,y)-form to x-form $(\star\star)$ under as weak as possible assumptions on the parameters, with a bounded operator as remainder.

4. Application to fractional order boundary problems.

For the fractional-order operators, we have generalized the a-transmission condition to $C^\tau S^m$ symbol classes and $C^{1+\tau}$-domains. It is in particular satisfied by $(-\Delta)^a$, and by its generalizations to ψdo's P that are of order $2a$ and **even**, i.e.,

$$p_j(x,-\xi) = (-1)^j p_j(x,\xi), \text{ all } j \in \mathbb{N}_0.$$

By localization and scaling, we obtain the basic regularity result:

Theorem. *Let $1 < p < \infty$, $0 < a < 1$, $\tau \geq 1$ with $\tau > 2a$. Let P be a strongly elliptic ψdo that is* **even** *with symbol in $C^\tau S^{2a}(\mathbb{R}^n \times \mathbb{R}^n)$, and let Ω be a bounded $C^{1+\tau}$-domain. When $u \in \dot{H}_p^a(\overline{\Omega})$ solves the homogeneous Dirichlet problem for P, then for $0 \leq s < \tau - 2a$,*

$$f \in \overline{H}_p^s(\Omega) \implies u \in \dot{H}_p^{s+2a}(\overline{\Omega}) + d^a \overline{H}_p^{s+a}(\Omega).$$

More precisely, u belongs to a specially defined a-transmission space $H^{a(s+2a)}(\overline{\Omega})$ contained in the right-hand side.
An important aspect of the result is that it "fills out" the range between C^∞-domains (G. '14/'15) and domains with low smoothness (Ros-Oton and Serra '14) in a way that has not been done before.

THANK YOU FOR YOUR ATTENTION!

Extra material:

Theorem. *Let* $\tau > 0$ *and* $m < \tau$.
Let $p \in C^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. *Under a* $C^{1+\tau}$*-diffeomorphism* $F\colon \mathbb{R}^n \to \mathbb{R}^n$ *such that* $c_0 \le |\det(\nabla F(x))| \le C_0$ *with* $c_0, C_0 > 0$, P *transforms to an operator* $\underline{P}$,

$$\underline{P} = F^* P F^{*,-1} = \mathrm{Op}(q(x,y,\xi)) + R_1,$$

where $q(x,y,\xi) \in C^\tau S^m_{1,0}(\mathbb{R}^{2n} \times \mathbb{R}^n)$ *and* $R_1\colon L_p(\mathbb{R}^n) \to H^s_p(\mathbb{R}^n)$ *for* $s < \min\{\tau, \tau - m\}$, $1 < p < \infty$.
Here for any nonnegative integer $l < \tau$,

$$\mathrm{Op}(q(x,y,\xi))u(x) = \sum_{|\alpha| \le l} \mathrm{Op}(p_\alpha(x,\xi))u(x) + \mathrm{Op}(r(x,y,\xi))u(x),$$

where $p_\alpha(x,\xi) = \frac{1}{\alpha!}\partial_y^\alpha D_\xi^\alpha q(x,y,\xi)|_{y=x} \in C^{\tau-|\alpha|} S^{m-|\alpha|}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, *and* $r(x,y,\xi) \in C^{\tau-l} S^{m-l}_{1,0}(\mathbb{R}^{2n} \times \mathbb{R}^n)$, *with* $r(x,x,\xi) = 0$. *The operators map continuously*

$$\mathrm{Op}(p_\alpha(x,\xi))\colon H_p^{s+m-|\alpha|}(\mathbb{R}^n) \to H^s_p(\mathbb{R}^n) \quad \text{for } |s| < \tau - |\alpha|,$$
$$\mathrm{Op}(r(x,y,\xi))\colon H_p^{(s+m-l)_+}(\mathbb{R}^n) \to H^s_p(\mathbb{R}^n) \quad \text{for } 0 \le s < \min\{\tau - l, \tau - m\}.$$