Hyperbolic systems with non-diagonalisable principal part

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Introduction

We consider

$$\begin{cases} D_t u = A(t,x,D_x)u + B(t,x,D_x)u + f(t,x), & (t,x) \in [0,T]\times\mathbb{R}^n, \\ u|_{t=0} = u_0, & x \in \mathbb{R}^n, \end{cases} \quad (1)$$

where $n \geq 1$, $m \geq 2$ and $D_t = -\mathrm{i}\partial_t$, $D_x = -\mathrm{i}\partial_x$. We assume that $A(t,x,D_x) = \big(a_{ij}(t,x,D_x)\big)_{i,j=1}^m$ is an $m\times m$ matrix of continuously time dependent pseudo-differential operators of order 1, i.e., $a_{ij} \in C([0,T],\Psi^1_{1,0}(\mathbb{R}^n))$ and that $B(t,x,D_x) = \big(b_{ij}(t,x,D_x)\big)_{i,j=1}^m$ is an $m\times m$ matrix of pseudo-differential operators of order 0, i.e., $b_{ij} \in C([0,T],\Psi^0_{1,0}(\mathbb{R}^n))$. We also assume that the matrix A is upper triangular and hyperbolic, i.e.,

$$\begin{aligned} A(t,x,D_x) &= \Lambda(t,x,D_x) + N(t,x,D_x) \\ &= \mathrm{diag}(\lambda_1(t,x,D_x),\lambda_2(t,x,D_x),\ldots,\lambda_m(t,x,D_x)) + N(t,x,D_x) \end{aligned}$$

with real eigenvalues $\lambda_1(t,x,\xi),\lambda_2(t,x,\xi),\ldots,\lambda_m(t,x,\xi)$ of $A(t,x,\xi)$ and

$$N(t,x,D_x) = \begin{pmatrix} 0 & a_{12}(t,x,D_x) & a_{13}(t,x,D_x) & \cdots & a_{1m}(t,x,D_x) \\ 0 & 0 & a_{23}(t,x,D_x) & \cdots & a_{2m}(t,x,D_x) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \ldots & a_{m-1m}(t,x,D_x) \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

Furthermore, we introduce the following two hypotheses:

(H1) For the coefficients of the lower order term $B(t,x,D_x)$:

the lower order terms b_{ij} belong to $C([0,T],\Psi^{j-i})$ for $i > j$.

(H2) For some theorems, we assume that A does not depend on t, i.e. $A = A(x,D_x)$ and satisfies: there exists $M \in \mathbb{N}$ such that if $\lambda_j(x,\xi) = \lambda_k(x,\xi)$ for some $j,k \in \{1,\ldots,m\}$ and $\lambda_j(x,\xi)$ and $\lambda_k(x,\xi)$ are not identically equal near (x,ξ) then there exists some $N \leq M$ such that

$$\lambda_j(x,\xi) = \lambda_k(x,\xi) \;\Rightarrow\; H^N_{\lambda_j}(\lambda_k) := \{\lambda_j,\{\lambda_j,\ldots,\{\lambda_j,\lambda_k\}\}\ldots\}(x,\xi) \neq 0, \quad \text{(H2)}$$

where the Poisson bracket $\{\cdot,\cdot\}$ in $H^N_{\lambda_j}$ is iterated N times.

Well-posedness

In Garetto et al. [2018], we prove a well-posedness result for (1) under hypothesis (H1):

Theorem 1. *Consider the Cauchy problem* (1)*, where $A(t,x,D_x)$ and $B(t,x,D_x)$ are as described in the introduction and $B(t,x,D_x)$ satisfies (H1). If now $u^0_k \in H^{s+k-1}(\mathbb{R}^n)$ and $f_k \in C([0,T],H^{s+k-1})$ for $k = 1,\ldots,m$, then* (1) *has a unique anisotropic Sobolev solution u, i.e., $u_k \in C([0,T],H^{s+k-1})$ for $k = 1,\ldots,m$.*

This theorem is proved by making use of the triangular form, solving the last equation and then iteratively building the solution of the system from the solutions to scalar equations. For each characteristic λ_j of A, we denote by $G^0_j\theta$ and $G_j g$ the respective solution solution to

$$\begin{cases} D_t w = \lambda_j(t,x,D_x)w + b_{jj}(t,x,D_x)w, \\ w(0,x) = \theta(x), \end{cases} \text{ and } \begin{cases} D_t w = \lambda_j(t,x,D_x)w + b_{jj}(t,x,D_x)w + g(t,x), \\ w(0,x) = 0. \end{cases}$$

The operators G^0_j and G_j can be microlocally represented by Fourier integral operators

$$G^0_j\theta(t,x) = \int e^{\mathrm{i}\varphi_j(t,x,\xi)} a_j(t,x,\xi)\widehat{\theta}(\xi)d\xi$$

and

$$G_j g(t,x) = \int_0^t \int e^{\mathrm{i}\varphi_j(t,s,x,\xi)} A_j(t,s,x,\xi)\widehat{g}(s,\xi)d\xi ds = \int_0^t \mathcal{E}_j(t,s)g(s,x)ds,$$

where

$$\mathcal{E}_j(t,s)g(s,x) = \int e^{\mathrm{i}\varphi_j(t,s,x,\xi)} A_j(s,x,\xi)\widehat{g}(s,\xi)d\xi$$

with $\varphi_j(t,s,x,\xi)$ solving the eikonal equation

$$\begin{cases} \partial_t\varphi_j = \lambda_j(t,x,\nabla_x\varphi_j), \\ \varphi_j(s,s,x,\xi) = x\cdot\xi, \end{cases} \quad \varphi_j(t,x,\xi) := \varphi_j(t,0,x,\xi).$$

The amplitudes $A_{j,-k}(t,s,x,\xi)$ of order $-k$, $k\in\mathbb{N}$, giving $A_j \sim \sum_{k=0}^\infty A_{j,-k}$, and they satisfy the usual transport equations with initial data at $t = s$, and we have $a_j(t,x,\xi) = A_j(t,0,x,\xi)$.

The components of the solution u of (1) is given by a composition of the operators described above together with principal part coefficients and lower order coefficients. That is where hypothesis (H1) comes into play. For example in the case $m = 2.$, we get

$$\begin{cases} u_1 = U^0_1 + G_1((a_{12}+b_{12})u_2), \\ u_2 = U^0_2 + G_2(b_{21}u_1), \end{cases} \quad U^0_j = G^0_j u^0_j + G_j(f_j), \quad j = 1,2.$$

That then gives

$$\begin{aligned} u_1 &= \tilde{U}^0_1 + G_1(a_{12}G_2(b_{21}u_1)) + G_1(b_{12}G_2(b_{21}u_1)) \\ \tilde{U}^0_1 &= G^0_1 u^0_1 + G_1(f_1) + G_1((a_{12}+b_{12})U^0_2). \end{aligned}$$

One then gets to the final result by setting up a fixed point problem to which Banach's fixed point theorem can be applied. A general time interval $[0,T]$ can be iteratively covered since the estimates involved for the G's do only depend on the coefficients and not the initial data.

Solution representations and regularity results

In the case of $A = A(x,D_x)$, asking in addition to (H1) on the lower order terms also (H2) for the principal part, the solutions of (1) can be represented explicitly modulo some smoothing operators. Here, we state the principal results and refer to Garetto et al. [2020] for the details and proofs.

Theorem 2. *Consider* (1) *with $A = A(x,D_x)$ and $B(t,x,D_x)$ satisfying properties described above and let in addition (H1) and (H2) be satisfied. Let u_0 and f have components u^0_j and f_j, respectively, with $u^0_j \in H^{s+j-1}(\mathbb{R}^n)$ and $f_j \in C([0,T],H^{s+j-1})$ for $j = 1,\ldots,m$. Then, for any $N\in\mathbb{N}$, the components u_j, $j = 1,\ldots,m$, of the solution u are given by*

$$u_j(t,x) = \sum_{l=1}^m \Big(\mathcal{H}^{l-j}_{j,l}(t) + R_{j,l}(t)\Big)u^0_l + \Big(\mathcal{K}^{l-j}_{j,l}(t) + S_{j,l}(t)\Big)f_l,$$

where $R_{j,l}$, $S_{j,l} \in \mathcal{L}(H^s, C([0,T],H^{s+N-l+j}))$ and the operators $\mathcal{H}^{l-j}_{j,l}$, $\mathcal{K}^{l-j}_{j,l} \in \mathcal{L}(C([0,T],H^s), C([0,T],H^{s-l+j}))$ are integrated Fourier Integral Operators of order $l-j$.

Using the explicit solution representations, we get

Theorem 3. *Let $p \in (1,\infty)$ and $\alpha = (n-1)\left|\frac{1}{p}-\frac{1}{2}\right|$. Consider* (1) *under (H1) and (H2). Then, for any compactly supported $u_0 \in L^p_\alpha \cap L^2_{comp}$, the solution $u = u(t,x)$ of the Cauchy problem* (1) *satisfies $u(t,\cdot) \in L^p_{loc}$, for all $t\in[0,T]$. Moreover, there is a positive constant C_T such that*

$$\sup_{t\in[0,T]} \|u(t,\cdot)\|_{L^p_{loc}} \leq C_T\|u_0\|_{L^p_\alpha}.$$

Local estimates can be obtained in other spaces as well, for $s\in\mathbb{R}$ and α as above. In detail, assuming u_0 below is compactly supported, we have that $u_0 \in L^p_{s+\alpha}$ implies $u(t,\cdot)\in L^p_s$; that $u_0 \in C^{s+\frac{n-1}{2}}$ implies $u(t,\cdot)\in C^s$; and, for $1 < p \leq q \leq 2$, that $u_0 \in L^p_{s-\frac{1}{q}+\frac{n}{p}-\frac{n-1}{2}}$ implies $u(t,\cdot)\in L^q_s$.

Propagation of singularities

Operators of the form $(\prod_\sigma G_{\sigma(i)})^k$, which appear in the solution representation above in the operators $\mathcal{H}^{l-j}_{j,l}$, are of the general form

$$Q_l = \int_0^t \int_0^{t_1} \cdots \int_0^{t_{l-1}} D(\bar{t})H(\bar{t})\,dt_l\ldots dt_1, \quad H(\bar{t}) = e^{i\lambda_{j_1}t_1}e^{i\lambda_{j_2}(t_2-t_1)}\cdot\ldots\cdot e^{i\lambda_{j_l}(t_l-t_{l-1})}e^{-i\lambda_{j_l}t_l}.$$

For these operators, we have $Q_l \in \mathcal{L}(H^s, H^{s+N(l)})$, where $N(l)\to+\infty$ as $l\to+\infty$. The singularities propagate along broken Hamiltonian flows.

Let $J = \{j_1,\ldots,j_{l+1}\}$, $1\leq j_k\leq m$, $j_k\neq j_{k+1}$. From the definition of $H(\bar{t})$, we have that its canonical relation $\Lambda^{\bar{t}} \subseteq T^*\mathbb{R}^n\times T^*\mathbb{R}^n$ is given by

$$\Lambda^{\bar{t}} = \left\{(x,p,y,\xi)\,:\,(x,p) = \Psi^{\bar{t}}(y,\xi)\right\}, \quad \Psi^{\bar{t}} = \Phi^{t_1}_{j_1}\circ\cdots\circ\Phi^{t_l-t_{l-1}}_{j_l}\circ\Phi^{-t_l}_{j_{l+1}}$$

and the Φ^t_j are the transformations corresponding to a shift by t along the trajectories of the Hamiltonian flow defined by the λ_j.

Let $\Phi_J(t,x,\xi)$ be the corresponding broken Hamiltonian flow. It means that points follow bicharacteristics of λ_{j_1} until meeting the characteristic of λ_{j_2}, and then continue along the bicharacteristic of λ_{j_2}, etc.

Operators of the type Q_l can be rewritten as standard Fourier integral operators where the domain of integration is not the whole space but a simplex. For those operators, following arguments of Hörmander, we get $WF(Iu)\subset\bigcup_j\Lambda_j(WF(u))$. For details see Kamotski and Ruzhansky [2007], Garetto et al. [2020].

Corollary 1. *Let $n\geq 1$, $m\geq 2$, and consider* (1) *with $A = A(x,D_x)$ under hypotheses (H1) and (H2). Then, see above, we have an explicit representation of the solution u. Consequently, up to any Sobolev order (depending on N), the wave front set of u_j is given by*

$$WF(u_j(t,\cdot)) \subset \left(\bigcup_{l=1}^m WF(\mathcal{H}^{l-j}_{j,l}(t)u^0_l)\right)\bigcup\left(\bigcup_{l=1}^m WF(\mathcal{K}^{l-j}_{j,l}(t)f_l)\right), \quad (2)$$

with each of the wave front sets for terms in the right hand side of (2) *given by the propagation along the broken Hamiltonian flow as described above.*

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