Spectral analysis near a Dirac type crossing in a weak non-constant magnetic field (after H.D. Cornean, B. Helffer and R. Purice).

Bernard Helffer,
Laboratoire de Mathématiques Jean Leray,
Université de Nantes.
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Abstract

This is the continuation of a series of works by the three authors devoted to the justification of the Peierls substitution in the case of a weak magnetic field. Here we deal with two $2d$ Bloch eigenvalues which have a conical crossing. It turns out that in the presence of an almost constant weak magnetic field, the spectrum near the crossing develops gaps which remind of the Landau levels of an effective mass-less magnetic Dirac operator. This involves the semi-classical analysis for the Peierls-Onsager effective Hamiltonian which is done through the combination of different pseudo-differential calculi.

On Peierls-Onsager substitution

Let us briefly recall that the *Peierls-Onsager substitution* is used by physicists (Peierls, Luttinger) in the study of non-interacting electrons in a periodic potential (describing the lattice of atoms in the solid) and subjected to a magnetic field.
In the absence of a magnetic field, the periodic Hamiltonian is described in the Floquet representation as the sum of a countable family of multiplication operators living in some finite dimensional subspaces and given by some real functions $\{\lambda_n : \mathcal{B} \to \mathbb{R}\}_{n\in\mathbb{N}}$ defined on the Brillouin domain $\mathcal{B}$; these are the Bloch functions.
We recall that the Brillouin domain is the unit cell in the momentum space with respect to the dual of the lattice defined by the periodic potential. The Peierls-Onsager substitution consists in replacing the complete Hamiltonian in a magnetic field $B = dA$ by the effective Hamiltonian obtained by replacing the functions $\lambda_n(\theta)$ with $\theta \in \mathcal{B}$ by $\lambda_n\big(\theta - A(x)\big)$.

The function $(\theta, x) \mapsto \lambda_n(\theta - A(x))$ can actually be considered as the symbol of a pseudo-differential operator on $\mathbb{R}^2$.
Giving a mathematical meaning to these operators is not quite evident and a rich literature has been devoted to this subject.
Some important restrictive hypothesis imposed in these studies have been the existence of isolated Bloch bands (i.e. some function λ_{n_0} that does not intersect with any other), the existence of Wannier basis for such isolated Bloch bands and the constancy of the magnetic field.

An important difficulty in using the Peierls-Onsager effective Hamiltonians for obtaining a detailed spectral information comes from the presence of the Bloch eigenprojections and the fact that they live on subspaces that depend on the magnetic field.

In our previous work [CHP-1]-[CHP-2], we have considered a 2-dimensional situation in which we can allow for some slow variation of the intensity of the magnetic field and prove a rather detailed spectral analysis of the effective Hamiltonians.

First, in [CHP-1] we studied the bottom of the magnetically perturbed spectrum in a narrow window around the non-degenerate minimum of an isolated Bloch energy whose corresponding spectral projection had a zero Chern number and admitted an exponentially localized Wannier basis.

Later on, in [CHP-2] we generalized these results to situations in which the unperturbed bottom of the spectrum comes from a single Bloch eigenvalue which either might cross with others outside the narrow window, or its corresponding spectral subspace has a non-trivial topology.

Our general strategy is to isolate some simple effective Hamiltonian that on a small neighborhood of some point in the Brillouin domain approximates well the exact one in the absence of magnetic field.

We have in view

- either a minimum of a Bloch eigenvalue, where we use the quadratic form given by the Hessian of the given Bloch function,
- or a conical crossing point where we use a 2×2-matrix valued Dirac type Hamiltonian defined by the two crossing Bloch functions and their 1-dimensional eigenprojections.

The magnetic field that we considered is of the form

$$B_{\epsilon,\kappa}(x) = \epsilon B^\circ + \epsilon\kappa B(\epsilon x)$$

where B° is a constant magnetic field producing some spectral gaps controlled by $\epsilon \in [0, \epsilon_0]$ for some $\epsilon_0 > 0$ small enough and $\epsilon\kappa B(\epsilon x)$ is a slowly varying magnetic field considered as a perturbation controlled by $\kappa \in [0, \kappa_0]$ for some $\kappa_0 \in (0, 1]$.

Our aim is to show that in a neighborhood of the special spectral point corresponding to a conical crossing of two Bloch functions, the above magnetic field produces a family of spectral gaps with widths and separation controlled by ϵ and κ.

The periodic Hamiltonian.

We work in $\mathbb{R}^2$ in which a regular lattice Γ is given.
(One can take $\Gamma = \mathbb{Z}^2$ as basic example).

Definition

We consider the Γ-periodic functions $V^\Gamma \in BC^\infty(\mathbb{R}^2;\mathbb{R})$ and $A^\Gamma \in BC^\infty(\mathbb{R}^2;\mathbb{R}^2)$ and define the 2-dimensional Γ-periodic Hamiltonian H_Γ as the self-adjoint extension in $L^2(\mathbb{R}^2)$ of the symmetric operator

$$-\Delta_{A^\Gamma} + V^\Gamma : \mathcal{S}(\mathbb{R}^2) \to \mathcal{S}(\mathbb{R}^2)\,, \tag{1}$$

with

$$-\Delta_{A^\Gamma} := \sum_{j=1,2} \big(-i\partial_{x_j} - A_j^\Gamma(x)\big)^2\,.$$

We recall that the periodic operators on $L^2(\mathbb{R}^2)$ admit a kind of 'partial diagonalization' given by the Bloch-Floquet unitary map (see also section XIII.16 of [RS-4]). We define the Bloch-Floquet-Zak transform of a test function $\phi \in \mathcal{S}(\mathbb{R}^2)$ for $x \in \mathbb{R}^2$, $\theta \in \mathbb{R}^2$,

$$(\check{\mathcal{U}}_\Gamma \phi)(x,\theta) := \sum_{\gamma \in \Gamma} e^{-2\pi i[(x_1-\gamma_1)\theta_1+(x_2-\gamma_2)\theta_2]}\phi(x-\gamma)\,. \qquad (2)$$

We notice that for any $\phi \in \mathcal{S}(\mathbb{R}^2)$ we have the following behavior of its Bloch-Floquet transform:

$$\forall \alpha \in \Gamma : \quad \big(\check{\mathcal{U}}_\Gamma \phi\big)(x+\alpha,\theta) = \big(\check{\mathcal{U}}_\Gamma \phi\big)(x,\theta), \quad \forall (x,\theta) \in \mathbb{R}^2 \times \mathbb{R}^2, \tag{3}$$

$$\forall \nu \in \mathbb{Z}^2 : \quad \big(\check{\mathcal{U}}_\Gamma \phi\big)(x,\theta+\nu) = e^{-2\pi i(x_1\nu_1+x_2\nu_2)} \big(\check{\mathcal{U}}_\Gamma \phi\big)(x,\theta). \tag{4}$$

Due to the periodicity in the x-variable, we can project this variable on the 2-dimensional torus $\mathbb{T}$ and consider functions defined on $\mathbb{T} \times \mathbb{R}^2$. We restrict the variable $\theta \in \mathbb{R}^2$ to the square $Q := (-1/2, 1/2) \times (-1/2, 1/2)$.

The transformation $\check{\mathcal{U}}_\Gamma$ defines a unitary operator $L^2(\mathbb{R}^2) \overset{\sim}{\to} L^2(\mathbb{T}) \otimes L^2(Q)$.

In this representation H_Γ becomes the operator of multiplication with an operator-valued function of $\theta \in \mathbb{R}^2$ taking values self-adjoint operators $\check{H}(\theta)$ acting in $L^2(\mathbb{T})$.

Proposition

The operators $\check{H}(\theta)$ with $\theta \in \mathbb{R}^2$ are self-adjoint, lower semi-bounded with compact resolvent in $L^2(\mathbb{T})$ and their eigenvalues $\{\lambda_k(\theta)\}_{k\in\mathbb{N}}$ (also called the Bloch eigenvalues) in increasing order taking into account their multiplicity.

The magnetic field.

We are interested in exhibiting a structure of gaps created in the band spectrum of H_Γ by a weak constant magnetic field and in studying their stability when perturbing the magnetic field by a smaller bounded smooth magnetic field.
Given $(\epsilon, \kappa) \in [0,1] \times [0,1]$, the magnetic field as the form

$$B_{\epsilon,\kappa}(x) = \epsilon B^\circ + \epsilon\kappa B(x)\,, \tag{5}$$

where B° is a constant magnetic field that we shall take to be positive and $\kappa B(x)$ is a weak magnetic field considered has a perturbation of B°.

Let us choose some smooth vector potentials $A^\circ : \mathbb{R}^2 \to \mathbb{R}^2$ and $A : \mathbb{R}^2 \to \mathbb{R}^2$ such that:

$$B^\circ = \partial_1 A_2^\circ - \partial_2 A_1^\circ \,,\; B = \partial_1 A_2 - \partial_2 A_1 \,, \tag{6}$$

and

$$A^{\epsilon,\kappa}(x) := \epsilon A^\circ(x) + \kappa\epsilon A(x) \,,\; B_{\epsilon,\kappa} = \partial_1 A_2^{\epsilon,\kappa} - \partial_2 A_1^{\epsilon,\kappa} \,. \tag{7}$$

The vector potential A° is considered in the transverse gauge, i.e.

$$A^\circ(x) \;=\; (1/2)\big(-B^\circ x_2, B^\circ x_1\big). \tag{8}$$

We consider the following magnetic Schrödinger operator, that is essentially self-adjoint on $\mathcal{S}(\mathbb{R}^2)$:

$$H_\Gamma^{\epsilon,\kappa} := \Big(-i\partial_{x_1} - A_1^\Gamma(x) - A_1^{\epsilon,\kappa}(x)\Big)^2 + \Big(-i\partial_{x_2} - A_2^\Gamma(x) - A_2^{\epsilon,\kappa}(x)\Big)^2 + V^\Gamma(x) \tag{9}$$

and treat it as a perturbation of

$$H_\Gamma^{\epsilon} := \Big(-i\partial_{x_1} - A_1^\Gamma(x) + \epsilon B^\circ x_2/2\Big)^2 + \Big(-i\partial_{x_2} - A_2^\Gamma(x) - \epsilon B^\circ x_1/2\Big)^2 + V^\Gamma(x) \tag{10}$$

that is also essentially self-adjoint on $\mathcal{S}(\mathbb{R}^2)$.

Formulation of the main result.

Hypothesis H1

There exists a compact interval $I := [-\Lambda_-, \Lambda_+] \subset \mathbb{R}$ containing 0 in its interior, an index $k_0 \in \mathbb{N} \setminus \{0\}$, a point $\theta_0 \in Q$ and a compact neighborhood $\Sigma_I \subset Q$ of θ_0, diffeomorphic with the unit disk, such that:

$$
\begin{aligned}
& I \cap \lambda_k(\mathbb{T}_*) \neq \emptyset \Rightarrow k \in \{k_0, k_0 + 1\}, \\
& [-\Lambda_-, 0] = \lambda_{k_0}(\Sigma_I), \quad [0, \Lambda_+] = \lambda_{k_0+1}(\Sigma_I), \\
& \lambda_{k_0}(\theta) = \lambda_{k_0+1}(\theta) = 0 \Rightarrow \theta = \theta_0.
\end{aligned} \tag{11}
$$

For $\theta \in \Sigma_I$ we shall denote by

$$\lambda_-(\theta) := \lambda_{k_0}(\theta), \qquad \lambda_+(\theta) := \lambda_{k_0+1}(\theta). \tag{12}$$

We now express the nature of the touching of λ_- and λ_+ at θ_0 , the so-called conical crossing type.

Hypothesis H2

The map $\Sigma_I \ni \theta \mapsto \lambda_-(\theta)\lambda_+(\theta)$ has a non-degenerate maximum value equal to zero at θ_0.

We need one more notation. For any two subsets M_1, M_2 in a metric space $(\mathcal{M}, d)$ we denote by

$$d_{\mathrm{H}}(M_1, M_2) := \max\left\{ \sup_{x\in M_1} \inf_{y\in M_2} d(x,y)\,,\ \sup_{x\in M_2} \inf_{y\in M_1} d(x,y) \right\} \quad (13)$$

their *Hausdorff distance*.

Main Theorem

Under Hypotheses H1 and H2, let $H_\Gamma^{\epsilon,\kappa}$ be the magnetic Hamiltonian with a magnetic field $B^{\epsilon,\kappa}$ as above. Then there exists a self-adjoint operator $\mathfrak{L}$ acting on $L^2(\mathbb{R})$ with discrete spectrum $\sigma(\mathfrak{L})$ symmetric with respect to the origin, containing 0 and with all the eigenvalues of multiplicity 1, such that for any $L > 0$ situated in the middle of a gap of $(B^\circ)^{1/2}\sigma(\mathfrak{L})$, there exist positive ϵ_L, κ_L, and C_L such that for $0 < \epsilon \leq \epsilon_L$ and $\kappa \in [0, \kappa_L]$, we have

$$d_{\mathrm{H}}\left(\sigma\left(H_\Gamma^{\epsilon,\kappa}\right) \cap \left(-L\epsilon^{\frac{1}{2}}, L\epsilon^{\frac{1}{2}}\right), (\epsilon B^\circ)^{\frac{1}{2}}\sigma(\mathfrak{L}) \cap \left(-L\epsilon^{\frac{1}{2}}, L\epsilon^{\frac{1}{2}}\right)\right) \leq C_L\left(\sqrt{\kappa\epsilon} + \epsilon\right).$$

Remarks

- The set $(\epsilon B^\circ)^{\frac{1}{2}}\sigma(\mathfrak{L}) \cap \big(-L\epsilon^{\frac{1}{2}}, L\epsilon^{\frac{1}{2}}\big)$ consists of finitely many isolated points situated at a distance of order $\sqrt{\epsilon}$ from each other. Thus when both ϵ_L and κ_L are small enough, the set $\sigma\big(H_\Gamma^{\epsilon,\kappa}\big) \cap \big(-L\epsilon^{\frac{1}{2}}, L\epsilon^{\frac{1}{2}}\big)$ develops gaps of order $\sqrt{\epsilon}$, uniformly in κ.
- We only have to prove the main Theorem for $\kappa = 0$.

Perturbation Theorem ([CP-1] (2012))

If L is chosen as in Main Theorem, there exists a constant $C > 0$ such that

$$d_{\mathrm{H}}\Big(\sigma\big(H_\Gamma^{\epsilon,\kappa}\big) \cap (-L\sqrt{\epsilon}, L\sqrt{\epsilon}), \sigma\big(H_\Gamma^{\epsilon}\big) \cap (-L\sqrt{\epsilon}, L\sqrt{\epsilon})\Big) \le C\sqrt{\kappa\epsilon}.$$

An important difficulty in the proof of Main Theorem comes from the fact that while in [CHP-2] we were working near the bottom of the spectrum, using positivity conditions for invertibility, we are now working somewhere in the bulk of the spectrum, in the interval $I \subset \sigma\big(H_\Gamma\big)$.

Hence a new procedure for creating a spectral gap in the studied region inside the interval I with some stability with respect to the magnetic field perturbation has to be elaborated.

Moreover we have to replace the 1-dimensional smooth unit norm global section defining the quasi-band associated with λ_0 near its minimum with a smooth global orthonormal pair of sections having some good behavior with respect to the induced spectral gap.

Magnetic pseudo-differential calculus

One basic tool is the magnetic pseudo-differential calculus developed by the Roumanian school (V. Iftimie, R. Purice, M. Mantoiu).

Given some magnetic field B with an associated vector potential A with components of class $C^\infty_{\mathrm{pol}}(\mathbf{X};\mathbb{R})$, we can define a magnetic pseudodifferential calculus, $\forall(\Phi,\phi)\in\mathcal{S}(\mathbf{X}\times\mathbf{X}^*)\times\mathcal{S}(\mathbf{X})$,

$$\begin{aligned}&(\mathrm{Op}^A(\Phi)\phi)(x)\\ &\quad := (2\pi)^{-2}\int_{\mathbf{X}} dy \int_{\mathbf{X}^*} d\xi\, e^{i<\xi,(x-y)>} e^{-i\int_{[y,x]}A}\,\Phi\big((x+y)/2,\xi\big)\phi(y)\,.\end{aligned} \tag{14}$$

Similarly we can define a *'magnetic' Moyal product* $\Phi\,\sharp^B\,\Psi$ such that

$$\mathrm{Op}^A(\Phi\,\sharp^B\,\Psi) \,=\, \mathrm{Op}^A(\Phi)\circ\mathrm{Op}^A(\Psi). \tag{15}$$

One can prove that it only depends on the magnetic field B and not on the vector potential. Its properties, rather similar with those of the usual Moyal product are studied by Iftimie-Mantoiu-Purice in a series of papers.

We will now focus in this talk on one other aspect of the proof where two pseudo-differential calculus are involved:

- a global pseudo-differential calculus
- a semi-classical pseudodifferential calculus.

Semi-classical analysis for 2×2 systems

We present here the analog of [HS2](1990) in our more general context. We start from a semi-classical system of 2×2 matrix valued symbols and we take, for comparison with the standard semi-classical analysis, h as semi-classical parameter (called before ϵ) and the Weyl quantification.
We present results which were initially developed in [HS1-HS2] (1989-1990) in the context of the Harper's model. Although most of the analysis there is rather general, the case of "touching bands" is only solved in a particular situation and there is a need to detail some new aspects.

We consider a symbol in $S^0(\mathbb{R}^2, \mathcal{M}_{2\times 2}(\mathbb{C}))$

$$A(x,\xi,h) \sim A^0(x,\xi) + \sum_{j\geq 2} A^j(x,\xi) h^{\frac{j}{2}}\,, \tag{16}$$

$$A^*(x,\xi,h) = A(x,\xi,h)\,, \tag{17}$$

and assume that

$$A^0(0,0) = 0\,. \tag{18}$$

We consider $A^w(x, hD_x, h)$, the pseudodifferential operator whose Weyl symbol is $(x,\xi) \mapsto A(x, h\xi; h)$. $A^w(x, hD_x, h)$ is a bounded selfadjoint operator on $L^2(\mathbb{R})$.

Our aim is to analyze the spectrum $A^w(x, hD_x, h)$ in intervals of the form $(-Ch^{\frac{1}{2}}, Ch^{\frac{1}{2}})$.
We assume a degenerate crossing point situation, i.e. that the linear approximation of $A^0(x, \xi)$ at $(0, 0)$:

$$A_{lin}(x, \xi) = \begin{pmatrix} \alpha_{11}x + \beta_{11}\xi & \alpha_{12}x + \beta_{12}\xi \\ \alpha_{21}x + \beta_{21}\xi & \alpha_{22}x + \beta_{22}\xi \end{pmatrix},$$

has the property that there exists $C > 0$ such that

$$\textbf{H3} \quad -\det(A_{lin}(x, \xi)) \geq \frac{1}{C}(x^2 + \xi^2), \forall (x, \xi) \in \mathbb{R}^2. \qquad (19)$$

We will now consider two cases.

- The case when $A_0(x, \xi)$ is uniformly invertible outside of $(0, 0)$
- The case when $A(x, \xi, h)$ is $\Gamma \times \Gamma_*$ periodic and $A_0(x, \xi)$ is invertible outside $\Gamma \times \Gamma_*$

The first case will be called the one-crossing point case
The second one will be called the periodic case.

A typical example of the second situation was met in [HS2] (Harpers model with flux $\frac{1}{2}$) :

$$A_0(x,\xi) = \begin{pmatrix} \sin x & \sin\xi \\ \sin\xi & -\sin x \end{pmatrix} \text{ with associated } A_{lin}(x,\xi) = \begin{pmatrix} x & \xi \\ \xi & -x \end{pmatrix}.$$

All the models in [HS2] have the property that $\mathrm{Tr} A_0(x,\xi) = 0$ and other nice properties which are not satisfied here.
This permits a deeper analysis but we can not make a direct application of [HS2] for analyzing the first case.

Analysis of the linearized operator

Under Assumption **H3** there exists $B(x,\xi) \in S_{is}^{-1}(\mathbb{R}^2, \mathbb{C}^2)$ such that

$$B(x,\xi) = A_{lin}(x,\xi)^{-1} \text{ for } x^2 + \xi^2 \geq 1\,. \tag{20}$$

Here, for $p \in \mathbb{R}$, $S_{is}^{p}(\mathbb{R}^2, \mathbb{C})$ is the class of symbols a

$$|D_x^\ell D_\xi^m a(x,\xi)| \leq C_{\ell m}(1 + x^2 + \xi^2)^{\frac{p-\ell-m}{2}} \text{ for } (x,\xi) \in \mathbb{R}^2\,.$$

The theory of globally elliptic operators is presented in [He-84] (mainly on the basis of results with D. Robert or on results of Shubin).

From this theory we obtain

Theorem

$A_{lin}(x, D_x)$ is an essentially selfadjoint operator starting from $\mathcal{S}(\mathbb{R}, \mathbb{C}^2)$ and its closure $\mathcal{A} = \overline{A_{lin}(x, D_x)}$ as an unbounded selfadjoint operator in $L^2(\mathbb{R}; \mathbb{C}^2)$ has a compact resolvent and hence a discrete sequence of eigenvalues. In addition

1. Its domain is $B^1(\mathbb{R}, \mathbb{C}^2) = \{u \in L^2(\mathbb{R}; \mathbb{C}^2), xu \in L^2, u' \in L^2\}$.
2. All the eigenfunctions are in $\mathcal{S}(\mathbb{R}, \mathbb{C}^2)$.
3. The eigenvalues have multiplicity one.
4. If λ is an eigenvalue $-\lambda$ is an eigenvalue.

More on the spectrum

One can see that 0 belongs to the spectrum of $\mathcal{A}$. Here is direct proof by construction of the corresponding eigenfunction. By a symplectic transformation, we can add the assumption that

$$\det A_{lin}(x,\xi) = -a\,(\xi^2 + x^2)\,. \tag{21}$$

We then an element in the kernel in the form

$$\Phi(x) = e^{-\frac{x^2}{2}} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

with $(u_1, u_2) \in \mathbb{C}^2$.

Quasimodes near $(0,0)$

Performing a dilation of size $h^{\frac{1}{2}}$, it is easier to analyze

$$A^w(h^{\frac{1}{2}}x, h^{\frac{1}{2}}D_x, h)\,.$$

Then we follow what was done for the so called harmonic approximation in the case of the Schrödinger operator $-h^2\frac{d^2}{dx^2} + V(x)$.

Nevertheless we are performing here a "linear" approximation. We consider the formal Taylor expansion in (x,ξ) at $(0,0)$ and reexpress this expansion in powers of $h^{\frac{1}{2}}$. This leads to

$$A(h^{\frac{1}{2}}x, h^{\frac{1}{2}}\xi, h) \sim h^{\frac{1}{2}}A_{lin}(x,\xi)) + \sum_{\ell\in\mathbb{N},\ell>1} \hat{A}^\ell(x,\xi)\,h^{\frac{\ell}{2}}\,, \qquad (22)$$

where $\hat{A}^\ell(x,\xi)$ is a polynomial of degree ℓ.

Proposition

There exist two infinite sequences λ_ℓ and $u_\ell \in \mathcal{S}(\mathbb{R})$ with (λ_1, u_0) as above such that, for any k, we have

$$(A^w(x, hD_x, h) - \lambda^{(k)}(h))u^{(k-1)}(x, h) = \mathcal{O}(h^{\frac{k+1}{2}})\,, \tag{23}$$

in $L^2(\mathbb{R})$ with

$$\lambda^{(k)}(h) = \sum_{1\leq\ell\leq k} h^{\ell/2}\lambda_\ell\,,\; u^{(k-1)}(x, h) = h^{-\frac{1}{4}} \sum_{0\leq\ell\leq k-1} h^{\ell/2} u_\ell(h^{-\frac{1}{2}}x)\,.$$

As a corollary we obtain that for any $k \geq 1$, there exists $h_k > 0$ and C_k such that

$$\mathrm{dist}(\sigma(A^w(x, hD_x, h)), \lambda^{(k)}(h)) \leq C_k h^{\frac{k+1}{2}}\,, \forall h \in (0, h_k]\,. \tag{24}$$

Quasiresolvents in the one crossing point situation

The construction is obtained by combining the resolvent of the linearized operator and far from the crossing point the "elliptic" semi-classical inverse.
In this construction, we are obliged to mix the standard semi-classical calculus and the globally isotropic global pseudo-differential calculus.

The periodic case

For this part, we simply refer to Helffer-Sjöstrand [HS1]. We have determined the spectrum in $(-Ch^{\frac{1}{2}}, Ch^{\frac{1}{2}})$ of the "one crossing point" model (the crossing point being at $(0,0)$). For our periodic model it is enough, according to [HS1] to construct a family of one micro-well operator indexed by the lattice. For this we eliminate all the crossing points except $(0,0)$ by using a technique of "destruction of the crossing" given in the neighborhood of each point of $\Gamma \times \Gamma_* \setminus \{0,0\}$.

It is then proven in [HS1] that the spectrum is contained in a $\mathcal{O}(h^\infty)$ neighborhood of the one crossing point model. Hence we get:

Theorem

Let $A(x,\xi,h)$ a periodic symbol in $S^0(\mathbb{R}^2,\mathbb{C}^2)$ over the lattice $\Gamma \times \Gamma_*$ whose linearized symbol at $(0,0)$ satisfies **H3**. Assume that $A_0(x,\xi)$ is invertible for any $(x,\xi) \in \mathbb{R}^2 \setminus \{\Gamma \times \Gamma_*\}$. Then, for any $C > 0$, there exists h_0 and C_0 such that $\forall h \in (0,h_0]$,

$$\mathrm{dist}(\sigma(A^w(x,hD_x,h)) \cap (-Ch^{\frac{1}{2}}, Ch^{\frac{1}{2}}), h^{\frac{1}{2}}\sigma(\mathcal{A})) \leq C_0 h\,. \tag{25}$$

Combined with the construction with quasimodes we can localize in intervals of size $\mathcal{O}(h^\infty)$.

Thank you for your attention.

Bibliography.

N. Athmouni, M. M**u**antoiu, R. Purice: *On the continuity of spectra for families of magnetic pseudodifferential operators*. Journal of Mathematical Physics 51, 083517 (2010).

R. Beals: *Characterization of pseudodifferential operators and applications.* Duke Math. J. 44 (1), 45–57 (1977).

J. Bellissard: *C^* algebras in solid state physics. 2D electrons in a uniform magnetic field.* Operator algebras and applications **2**, 49–76, London Math. Soc. Lecture Note Ser., 136, Cambridge Univ. Press, Cambridge, (1988).

J. Bellissard: *Lipschitz continuity of gap boundaries for Hofstadter-like spectra*. Comm. Math. Phys. **160**(3), 599–613 (1994).

H. D. Cornean, B. Helffer, R. Purice: *Low lying spectral gaps induced by slowly varying magnetic fields*. Journal of Functional Analysis **273**(1), 206–282 (2017).

- H. D. Cornean, B. Helffer, R. Purice: *A Beals criterion for magnetic pseudo-differential operators proved with magnetic Gabor frames*. Comm. in P.D.E. **43**(8), 1196–1204 (2018).
- H. D. Cornean, B. Helffer, R. Purice: *Peierls' substitution for low lying spectral energy windows*. Journal of Spectral Theory **9**(4), 1179–1222 (2019).
- H. D. Cornean, I. Herbst, and G. Nenciu: *On the construction of composite Wannier functions*. Ann. H. Poincaré **17**, 3361–3398 (2016).
- H.D. Cornean, V. Iftimie, R. Purice: *Peierls substitution via minimal coupling and magnetic pseudo-differential calculus*. Reviews in Mathematical Physics **31**(3), 1950008 (2019).
- H.D. Cornean and D. Monaco: *On the construction of Wannier functions in topological insulators: the 3D case*. Ann. H. Poincaré **18**, 3863–3902 (2017).

H.D. Cornean, R. Purice: *On the regularity of the Hausdorff distance between spectra of perturbed magnetic Hamiltonians*. Operator Theory: Advances and Applications 224, 55–66 (2012).

H.D. Cornean, R. Purice: *Spectral edge regularity of magnetic Hamiltonians*. Journal of the London Mathematical Society, **92**(1), 89–104 (2015).

G. De Nittis and M. Lein: *Applications of magnetic ΨDO techniques to space-adiabatic perturbation theory*. Rev. Math. Phys. **23**(3), 233–260 (2011).

J. Dixmier: *Les algèbres d'opérateurs dans l'espace hilbertien*. (2-ème edition), Paris, Gauthier-Villars éditeur, (1969).

D. Fiorenza, D. Monaco, and G. Panati: *Construction of real-valued localized composite Wannier functions for insulators*. Ann. H. Poincaré **17**(1), 63–97 (2016).

- G. B. Folland: *Harmonic Analysis in Phase Space.* Princeton University Press, Princeton, New Jersey, (1989).
- S. Freund and S. Teufel: *Peierls substitution for magnetic Bloch bands*. Anal. & PDE **9**(4), 773–811 (2016).
- J. M. Gracia-Bondá, J. C. Vřilly: *Algebra of distributions suitable for phase-space Quantum Mechanics I.* J. Math. Phys. 29(4), 869–879 (1988).
- J. M. Gracia-Bondá, J. C. Vřilly: *Algebra of distributions suitable for phase-space Quantum Mechanics II. Topologies on the Moyal Algebra.* J. Math. Phys. 29(4), 880–887 (1988).
- A. Grossman, G. Loupias, E. M. Stein: *An algebra of pseudo-differential operators and Quantum Mechanics in phase space.* Ann. Inst. Fourier, 18(2), 343–368 (1969).
- S. J. Gustafson, I. M. Sigal: *Mathematical Concepts of Quantum Mechanics* (2-nd edition). Universitext Springer, (2011).

B. C. Hall: *Lie groups, Lie algebras, and representations : an elementary introduction*. Graduate Texts in Mathematics 222, Springer (2015).

B. Helffer: *Théorie spectrale pour des opérateurs globalement elliptiques*. Astérisque 112, Société Mathématique de France, (1984), X + 197 p.

B. Helffer and J. Sjöstrand: *Equation de Schrödinger avec champ magnétique et équation de Harper*, in LNP **345**, Springer-Verlag, Berlin, Heidelberg and New York, 118–197 (1989).

B. Helffer and J. Sjöstrand:
Analyse semi-classique pour l'équation de Harper (avec application à l'équation de Schrödinger avec champ magnétique).
Mémoire de la SMF, No 34 (1988).

B. Helffer and J. Sjöstrand:

Analyse semi-classique pour l'équation de Harper II (comportement semi-classique près d'un rationnel).
Mémoire de la SMF, No 40, Tome 118, Fasc.1 (1990).

B. Helffer and J. Sjöstrand:
On diamagnetism and the de Haas-van Alphen effect.
Annales de l'I.H.P. section A, **52**(4), 303–375 (1990).

Lars Hörmander: *The Weyl Calculus of Pseudo-differential Operators*, Comm. Pure Appl. Math. 32, 359–443, (1979).

Lars Hörmander: *The Analysis of Linear Partial Differential Operators, I: Distribution Theory and Fourier Analysis*, (2-nd edition) Springer-Verlag, New York, (1990).

Lars Hörmander: *The Analysis of Linear Partial Differential Operators III: Pseudo-Differential Operators*. Springer-Verlag Berlin Heidelberg, (2007), pag. VIII + 525.

D. Husemoller: *Fibre bundles*, 3rd edition. No. 20 in Graduate Texts in Mathematics. Springer-Verlag, New York, (1994).

- V. Iftimie, M. Mantoiu, R. Purice: *Magnetic pseudodifferential operators*. Publ. Res. Inst. Math. Sci. 43, 585–623 (2007).
- V. Iftimie, M. Mantoiu, R. Purice: *Commutator Criteria for Magnetic Pseudodifferential Operators*. Comm. Partial Diff. Eq. 35, 1058–1094 (2010).
- V. Iftimie, R. Purice: *The Peierls-Onsager Effective Hamiltonian in a complete gauge covariant setting: Determining the spectrum*, Journal of Spectral Theory, 5, 445–531 (2015).
- T. Kato: *Perturbation Theory for Linear Operators.* Springer-Verlag Berlin Heidelberg, (1995).
- W. Kirsch and B. Simon: *Comparison theorems for the gap of Schrödinger operators.* J. Funct. Anal. **75**(2), 396-410 (1987).
- P. Kuchment: *An overview of periodic elliptic operators.* Bull. Amer. Math. Soc. **53**, 343–414 (2016).

J.M. Luttinger: *The Effect of a Magnetic Field on Electrons in a Periodic Potential.* Phys. Rev. **84**, 814-817 (1951).

M. M**u**antoiu, R. Purice: *The magnetic Weyl calculus*. Journal of Mathematical Physics 45 (4), 1394–1417 (2004).

M. M**u**antoiu, R. Purice, and S. Richard: *Spectral and propagation results for magnetic Schrödinger operators; a C^*-Algebraic Approach*.
J. Funct. Anal. **250**, 42–67 (2007).

G. Nenciu: *Bloch electrons in a magnetic field, rigorous justification of the Peierls-Onsager effective Hamiltonian*.
Lett. Math.Phys. **17**, 247–252 (1989).

G. Nenciu: *Dynamics of Bloch electrons in electric and magnetic fields, rigorous justification of the effective Hamiltonians.*
Rev. Mod.Phys. **63** (1), 91–127 (1991).

- G. Nenciu: *On asymptotic perturbation theory for quantum mechanics: almost invariant subspaces and gauge invariant magnetic perturbation theory.* J. Math. Phys. **43**(3), 1273–1298 (2002).
- Howard Osborn: *Vector Bundles. Vol. 1: Foundations and Stiefel - Whitney classes*. Academic Press (1982).
- R.S. Palais: *Homotopy theory of infinite dimensional manifolds*. Topology **5**, 1–16 (1966).
- G. Panati, H. Spohn, and S. Teufel: *Effective dynamics for Bloch electrons: Peierls substitution and beyond*. Comm. Math. Phys. **242**, no. 3, 547–578 (2003).
- R.E. Peierls: *Quantum Theory of Solids*. Oxford University Press, (1955).
- R. Rammal and J. Bellissard: *An algebraic semi-classical approach to Bloch electrons in a magnetic field*. J. de Phys. **51**(17), 1803–1830 (1990).

M. Reed, B. Simon: *Methods of Modern Mathematical Physics*. Academic Press, Inc., San Diego, New York, Berkeley, Boston, London, Sydney, Tokyo, Toronto, (1972).

M. Reed, B. Simon: *Methods of Modern Mathematical Physics. Vol. IV: Analysis of operators*. Academic Press, Inc., San Diego, New York, Berkeley, Boston, London, Sydney, Tokyo, Toronto, (1978).

J. Sjöstrand: *Microlocal analysis for the periodic magnetic Schrödinger equation and related questions*. Microlocal Analysis and Applications. 227–332 (1989). DOI: 10.1007/BFb0085120

N. Steenrod: *The Topology of Fibre Bundles*. Princeton University Press, (1959).

M. E. Taylor: *Noncommutative Harmonic Analysis*. Mathematical Surveys and Monographs 22. American Mathematical Society, Providence, Rhode Island, (1986).

J. Zak: *Magnetic translation group*. Phys. Rev **134**, 1602 A, (1964).