Uncertainty principles and null-controllability of evolution equations enjoying Gelfand-Shilov smoothing effects

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Uncertainty principles

Uncertainty principles are **mathematical results** that give limitations on the simultaneous concentration of a **function** and its **Fourier transform**

The **Heisenberg's uncertainty principle** gives that for all $f \in L^2(\mathbb{R}^n)$, $1 \leq j \leq n$, $a, b \in \mathbb{R}$,

$$\Big(\int_{\mathbb{R}^n} (x_j - a)^2 |f(x)|^2 dx \Big) \Big(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (\xi_j - b)^2 |\widehat{f}(\xi)|^2 d\xi \Big) \geq \frac{1}{4} \|f\|^4_{L^2(\mathbb{R}^n)}$$

The above **inequality** is an **equality** if and only if f is of the **type**

$$f(x) = g(x_1, ..., x_{j-1}, x_{j+1}, ..., x_n) e^{-ibx_j} e^{-\alpha(x_j - a)^2}$$

with $g \in L^2(\mathbb{R}^{n-1})$, $\alpha > 0$

There are **various uncertainty principles** of **different nature**

Another formulation of **uncertainty principles** is that a **non-zero function** and its **Fourier transform** cannot both have small supports : for instance, a **non-zero** $L^2(\mathbb{R}^n)$**-function** whose **Fourier transform** is **compactly supported** must be an **analytic function** with a **discrete zero set** and therefore a **full support**

Annihilating pairs

This leads to the notion of **weak annihilating pairs** as well as the corresponding **quantitative notion** of **strong annihilating pairs**

Let S, Σ be two **measurable subsets** of $\mathbb{R}^n$:

- The **pair** (S, Σ) is said to be a **weak annihilating pair** if the only function $f \in L^2(\mathbb{R}^n)$ with $\operatorname{supp} f \subset S$ and $\operatorname{supp} \widehat{f} \subset \Sigma$ is **zero** $f = 0$
- The **pair** (S, Σ) is said to be a **strong annihilating pair** if there exists a positive constant $C = C(S, \Sigma) > 0$ such that

$$\forall f \in L^2(\mathbb{R}^n), \quad \int_{\mathbb{R}^n} |f(x)|^2 dx \leq C\Big(\int_{\mathbb{R}^n \setminus S} |f(x)|^2 dx + \int_{\mathbb{R}^n \setminus \Sigma} |\widehat{f}(\xi)|^2 d\xi\Big)$$

It can be checked that a **pair** (S, Σ) is a **strong annihilating pair** if and only if there exists a positive constant $D = D(S, \Sigma) > 0$ such that

$$\forall f \in L^2(\mathbb{R}^n), \ \operatorname{supp} \widehat{f} \subset \Sigma, \quad \|f\|_{L^2(\mathbb{R}^n)} \leq D\|f\|_{L^2(\mathbb{R}^n \setminus S)}$$

Some examples of annihilating pairs

As seen above, (S, Σ) is a **weak annihilating pair** if S and Σ are compact sets

More generally, **Benedicks** proved that (S, Σ) is a **weak annihilating pair** if S and Σ are **sets** of finite Lebesgue measure $|S|, |\Sigma| < +\infty$

More specifically, **Amrein** and **Berthier** proved that (S, Σ) is a **strong annihilating pair** if S and Σ are **sets** of finite Lebesgue measure $|S|, |\Sigma| < +\infty$

$$\forall f \in L^2(\mathbb{R}^n), \quad \int_{\mathbb{R}^n} |f(x)|^2 dx \leq C(S, \Sigma)\Big(\int_{\mathbb{R}^n \setminus S} |f(x)|^2 dx + \int_{\mathbb{R}^n \setminus \Sigma} |\widehat{f}(\xi)|^2 d\xi\Big)$$

In the case $n = 1$, **Nazarov** obtained the following **quantitative estimate**

$$C(S, \Sigma) \leq \kappa e^{\kappa |S||\Sigma|}$$

with $\kappa > 0$. This result was extended by **Jaming** in the **multi-dimensional case**

$$C(S, \Sigma) \leq \kappa e^{\kappa (|S||\Sigma|)^{1/n}}$$

when in addition **one of the two subsets** of **finite Lebesgue measure** is **convex**

An **exhaustive description** of **strong annihilating pairs** is for now **out of reach**

Logvinenko-Sereda Theorem

The **Logvinenko-Sereda theorem** provides a **complete description** of support sets S forming a **strong annihilating pair** with any bounded spectral set Σ

Let $S, \Sigma \subset \mathbb{R}^n$ be **measurable subsets** with Σ bounded. The following assertions are **equivalent** :

- The pair (S, Σ) is a **strong annihilating pair**
- The subset $\mathbb{R}^n \setminus S$ is **thick**, that is, there exists a **cube** $K \subset \mathbb{R}^n$ with sides parallel to coordinate axes and a positive constant $0 < \gamma \leq 1$ such that

$$\forall x \in \mathbb{R}^n, \quad |(K + x) \cap (\mathbb{R}^n \setminus S)| \geq \gamma |K| > 0$$

Notice that if (S, Σ) is a **strong annihilating pair** for some **bounded subset** Σ, then S defines a **strong annihilating pair** with **any bounded subset** Σ, but the above **constants** $C(S, \Sigma) > 0$ and $D(S, \Sigma) > 0$ **do depend** on Σ

In order to use the **Logvinenko-Sereda Theorem** in **control theory**, it is **essential** to understand how the **positive constant** $D(S, \Sigma) > 0$,

$$\forall f \in L^2(\mathbb{R}^n), \ \text{supp}\, \widehat{f} \subset \Sigma, \quad \|f\|_{L^2(\mathbb{R}^n)} \leq D(S, \Sigma) \|f\|_{L^2(\mathbb{R}^n \setminus S)}$$

depends on the **bounded set** Σ, when $\mathbb{R}^n \setminus S$ is a fixed **thick set**

Quantitative version of Logvinenko-Sereda Theorem

This question was answered by **Kovrijkine** (2001) : there exists a **universal positive constant** $C_n > 0$ depending only on the **dimension** $n \geq 1$ such that if $\tilde{S}$ is a γ-thick set at scale $L > 0$, that is,

$$\forall x \in \mathbb{R}^n, \quad |\tilde{S} \cap (x + [0, L]^n)| \geq \gamma L^n$$

with $0 < \gamma \leq 1$, then the **following estimate holds**

$$\forall R > 0, \forall f \in L^2(\mathbb{R}^n), \ \operatorname{supp} \widehat{f} \subset [-R, R]^n, \quad \|f\|_{L^2(\mathbb{R}^n)} \leq \left(\frac{C_n}{\gamma}\right)^{C_n(1+LR)} \|f\|_{L^2(\tilde{S})}$$

Thanks to this **quantitative version** of the **Logvinenko-Sereda Theorem**, **Egidi** and **Veselic**, and **Wang**, **Wang**, **Zhang** and **Zhang** independently proved that the **heat equation**

$$\begin{cases} (\partial_t - \Delta_x) f(t, x) = u(t, x) \mathbb{1}_\omega(x) \,, & x \in \mathbb{R}^n, t > 0, \\ f|_{t=0} = f_0 \in L^2(\mathbb{R}^n), & \end{cases}$$

is **null-controllable** in **any positive time** $T > 0$ from a **measurable control subset** $\omega \subset \mathbb{R}^n$ if and only if this subset ω is **thick** in $\mathbb{R}^n$

Null-controllability and observability

Let P be a **closed operator** on $L^2(\mathbb{R}^n)$ which is the **infinitesimal generator** of a **strongly continuous semigroup** $(e^{-tP})_{t\geq 0}$ on $L^2(\mathbb{R}^n)$, $T>0$ and ω be a **measurable subset** of $\mathbb{R}^n$. The equation

$$\begin{cases} (\partial_t + P)f(t,x) = u(t,x)\mathbb{1}_\omega(x)\,, & x \in \mathbb{R}^n, t>0, \\ f|_{t=0} = f_0 \in L^2(\mathbb{R}^n), \end{cases}$$

is **null-controllable** from the set ω in time $T>0$ if, for any **initial datum** $f_0 \in L^2(\mathbb{R}^n)$, there exists $u \in L^2((0,T)\times\mathbb{R}^n)$, supported in $(0,T)\times\omega$, such that the **mild (or semigroup) solution** satisfies $f(T,\cdot)=0$

By the **Hilbert Uniqueness Method**, the **null-controllability** is **equivalent** to the **observability** of the **adjoint system**

$$\begin{cases} (\partial_t + P^*)g(t,x) = 0\,, & x \in \mathbb{R}^n\,, \\ g|_{t=0} = g_0 \in L^2(\mathbb{R}^n), \end{cases}$$

that is, there exists a positive constant $C_T>0$ such that, for any **initial datum** $g_0 \in L^2(\mathbb{R}^n)$, the **mild (or semigroup) solution** satisfies

$$\int_{\mathbb{R}^n} |g(T,x)|^2 dx \leq C_T \int_0^T \Big(\int_\omega |g(t,x)|^2 dx \Big) dt$$

Abstract result of observability

The **necessity** of the **thickness property** for the **null-controllability** of the **heat equation** is a consequence of a **quasimodes construction** ; whereas the **sufficiency** can be derived from an **abstract observability result** obtained by **Beauchard-KPS** with contributions of **Miller** (2018) from an **adapted Lebeau-Robbiano method** :

Let Ω be an **open subset** of $\mathbb{R}^n$, ω be a **measurable subset** of Ω, $(\pi_k)_{k\in\mathbb{N}^*}$ be a family of **orthogonal projections** defined on $L^2(\Omega)$, $(e^{-tA})_{t\geq 0}$ be a **strongly continuous contraction semigroup** on $L^2(\Omega)$; $c_1, c_2, a, b, t_0, m_1 > 0$, $m_2 \geq 0$ with $a < b$. If the **spectral inequality**

$$\forall g \in L^2(\Omega), \forall k \geq 1, \quad \|\pi_k g\|_{L^2(\Omega)} \leq e^{c_1 k^a} \|\pi_k g\|_{L^2(\omega)}$$

and the **dissipation estimate**

$$\forall g \in L^2(\Omega), \forall k \geq 1, \forall 0 < t < t_0, \quad \|(1-\pi_k)(e^{-tA}g)\|_{L^2(\Omega)} \leq \frac{e^{-c_2 t^{m_1} k^b}}{c_2 t^{m_2}} \|g\|_{L^2(\Omega)}$$

hold, then the following **observability estimate holds**

$$\exists C > 1, \forall T > 0, \forall g \in L^2(\Omega), \ \|e^{-TA}g\|^2_{L^2(\Omega)} \leq C \exp\Big(\frac{C}{T^{\frac{am_1}{b-a}}}\Big) \int_0^T \|e^{-tA}g\|^2_{L^2(\omega)} dt$$

Null-controllability of the heat equation posed in $\mathbb{R}^n$

The **null-controllability** of the **heat equation** is derived while using **frequency cutoff operators** given by the **orthogonal projections** onto

$$E_k = \{f \in L^2(\mathbb{R}^n) : \ \text{supp } \widehat{f} \subset [-k, k]^n\}$$

The **dissipation estimate** follows from the **explicit formula**

$$(\widehat{e^{t\Delta_x} g})(t, \xi) = \widehat{g}(\xi) e^{-t|\xi|^2}, \quad t \geq 0, \ \xi \in \mathbb{R}^n$$

whereas the **spectral inequality** is given by the **quantitative formulation of the Logvinenko-Sereda theorem**

The **abstract result** of **null-controllability/observability** is applied with **parameters** $a = 1$ and $b = 2$ satisfying the **condition** $0 < a < b$

As there is a **gap** between the **cost of the localization** $(a = 1)$ given by the **spectral inequality** and its **compensation** by the **dissipation estimate** $(b = 2)$, we **could have expected** that the **null-controllability** of the **heat equation** could have held under **weaker assumptions** than the **thickness property**, by allowing **higher costs** for **localization** $(1 < a < 2)$, but the **Logvinenko-Sereda theorem** shows that it is actually **not the case**

Gelfand-Shilov regularity

The **abstract result** does not only apply with **frequency cutoff projections** and a **dissipation estimate** induced by **Gevrey type regularizing effects**

Other regularities than the **Gevrey regularity** can be taken into account as e.g. the **Gelfand-Shilov regularity**

The **Gelfand-Shilov spaces** $S^{\mu}_{\nu}(\mathbb{R}^n)$, with $\mu, \nu > 0$, $\mu + \nu \geq 1$, are defined as the spaces of smooth functions $f \in C^{\infty}(\mathbb{R}^n)$ satisfying the estimates

$$\exists A, C > 0, \quad \sup_{x \in \mathbb{R}^n} |x^{\beta} \partial_x^{\alpha} f(x)| \leq C A^{|\alpha|+|\beta|} (\alpha!)^{\mu} (\beta!)^{\nu}, \quad \alpha, \beta \in \mathbb{N}^n$$

These **Gelfand-Shilov spaces** $S^{\mu}_{\nu}(\mathbb{R}^n)$ may also be **characterized** as the spaces of **Schwartz functions** $f \in \mathscr{S}(\mathbb{R}^n)$ satisfying the estimates

$$\exists C > 0, \varepsilon > 0, \quad |f(x)| \leq C e^{-\varepsilon |x|^{1/\nu}}, \ x \in \mathbb{R}^n, \qquad |\widehat{f}(\xi)| \leq C e^{-\varepsilon |\xi|^{1/\mu}}, \ \xi \in \mathbb{R}^n$$

More generally, the **symmetric Gelfand-Shilov spaces** $S^{\mu}_{\mu}(\mathbb{R}^n)$, with $\mu \geq 1/2$, can be **characterized** through the **decomposition** into $(\Phi_{\alpha})_{\alpha \in \mathbb{N}^n}$ the **Hermite basis**

$$f \in S^{\mu}_{\mu}(\mathbb{R}^n) \Leftrightarrow f \in L^2(\mathbb{R}^n), \ \exists t_0 > 0, \ \left\| \langle f, \Phi_{\alpha} \rangle_{L^2} \exp\left(t_0 |\alpha|^{\frac{1}{2\mu}}\right) \right)_{\alpha \in \mathbb{N}^n} \right\|_{l^2(\mathbb{N}^n)} < +\infty$$

Quadratic operators

Quadratic operators are **pseudodifferential operators** defined in the **Weyl quantization**

$$q^w(x,D_x)u(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} q\Big(\frac{x+y}{2},\xi\Big)u(y)dyd\xi$$

by **symbols** $q(x,\xi)$ which are **complex-valued quadratic forms**. These operators are **non-selfadjoint differential operators** with simple and explicit expression as

$$\forall \alpha,\beta \in \mathbb{N}^n,\ |\alpha+\beta| = 2, \quad \mathrm{Op}^w(x^\alpha\xi^\beta) = \frac{x^\alpha D_x^\beta + D_x^\beta x^\alpha}{2}, \quad D_x = i^{-1}\partial_x$$

When $\mathrm{Re}\ q \geq 0$, the operator $q^w(x,D_x)$ equipped with the **domain**

$$D(q) = \{u \in L^2(\mathbb{R}^n) : q^w(x,D_x)u \in L^2(\mathbb{R}^n)\}$$

is **maximally accretive** and generates a **contraction semigroup** $(e^{-tq^w})_{t\geq 0}$ on $L^2(\mathbb{R}^n)$

Singular space

The **Hamilton map** $F \in M_{2n}(\mathbb{C})$ of the **quadratic operator** $q^w(x, D_x)$ is **uniquely** defined by the **identity**

$$q(x,\xi;y,\eta) = \langle Q(x,\xi),(y,\eta)\rangle = \sigma((x,\xi),F(y,\eta)), \qquad (x,\xi),(y,\eta) \in \mathbb{R}^{2n}$$

where σ is the **canonical symplectic form**

The **singular space** of a **quadratic operator** $q^w(x, D_x)$ was defined by Hitrik and KPS (2009) as the following **finite intersection** of kernels

$$S = \Big(\bigcap_{j=0}^{2n-1} \mathrm{Ker}\big[\mathrm{Re}\, F(\mathrm{Im}\, F)^j\big]\Big) \cap \mathbb{R}^{2n} \subset \mathbb{R}^{2n}$$

where F denotes the **Hamilton map** of its **Weyl symbol** q

Gelfand-Shilov regularizing properties of semigroups

Let $q^w(x, D_x)$ be a **quadratic operator** whose **Weyl symbol** has a **non-negative** real part $\mathrm{Re}\ q \geq 0$ and a **zero singular space** $S = \{0\}$

Then, the **contraction semigroup** $(e^{-tq^w})_{t\geq 0}$ on $L^2(\mathbb{R}^n)$ is **smoothing** in the **Gelfand-Shilov space** $S_{1/2}^{1/2}(\mathbb{R}^n)$ for **any positive time** $t > 0$

$$\exists C > 1, \exists t_0 > 0, \forall u \in L^2(\mathbb{R}^n), \forall \alpha, \beta \in \mathbb{N}^n, \forall 0 < t \leq t_0,$$

$$\|x^\alpha \partial_x^\beta (e^{-tq^w} u)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C^{1+|\alpha|+|\beta|}}{t^{\frac{2k_0+1}{2}(|\alpha|+|\beta|+2n+s)}} (\alpha!)^{1/2} (\beta!)^{1/2} \|u\|_{L^2(\mathbb{R}^n)}$$

and

$$\exists C_0 > 1, \exists t_0 > 0, \forall 0 \leq t \leq t_0, \quad \|e^{\frac{t^{2k_0+1}}{C_0}(-\Delta_x^2+|x|^2)} e^{-tq^w}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C_0$$

where s is a **fixed integer** verifying $s > \frac{n}{2}$ and where $0 \leq k_0 \leq 2n-1$ is the **smallest integer** satisfying

$$\Big(\bigcap_{j=0}^{k_0} \mathrm{Ker}\big[\mathrm{Re}\ F(\mathrm{Im}\ F)^j\big]\Big) \cap \mathbb{R}^{2n} = \{0\}$$

Hitrik, KPS & Viola (2018)

Null-controllability of hypoelliptic quadratic equations

Let $q^w(x, D_x)$ be a **quadratic operator** whose **Weyl symbol** has a **non-negative** real part $\mathrm{Re}\ q \geq 0$ and a **zero singular space** $S = \{0\}$

The contraction **semigroup** $(e^{-tq^w})_{t \geq 0}$ enjoys **Gelfand-Shilov regularizing effects** satisfying : $\exists C_0 > 1,\ \exists t_0 > 0,\ \forall t \geq 0,$

$$\forall k \geq 0, \forall f \in L^2(\mathbb{R}^n), \quad \|(1 - \pi_k)(e^{-tq^w} f)\|_{L^2(\mathbb{R}^n)} \leq C_0 e^{-\delta(t)k} \|f\|_{L^2(\mathbb{R}^n)}$$

with

$$\delta(t) = \inf(t, t_0)^{2k_0+1}/C_0 \geq 0, \quad t \geq 0, \quad 0 \leq k_0 \leq 2n - 1$$

where

$$\pi_k g = \sum_{\alpha \in \mathbb{N}^n, |\alpha| \leq k} \langle g, \Phi_\alpha \rangle_{L^2(\mathbb{R}^n)} \Phi_\alpha, \quad k \geq 0$$

denotes the **orthogonal projection** onto the $(k+1)^{\mathrm{th}}$ **first energy levels** of the **harmonic oscillator**

The **abstract result** of **observability** applies if the **control subset** ω satisfies a **spectral inequality** for **finite combinations** of **Hermite functions** of the type

$$\exists C > 1, \forall k \geq 0, \forall f \in L^2(\mathbb{R}^n), \quad \|\pi_k f\|_{L^2(\mathbb{R}^n)} \leq C e^{Ck^a} \|\pi_k f\|_{L^2(\omega)}$$

with $a < 1$

Uncertainty principles for Hermite functions

Let $(\Phi_\alpha)_{\alpha\in\mathbb{N}^n}$ be the **Hermite functions** and $\mathcal{E}_N = \text{Span}_{\mathbb{C}}\{\Phi_\alpha\}_{\alpha\in\mathbb{N}^n, |\alpha|\leq N}$

As the **Lebesgue measure** of the **zero set** of a **non-zero analytic function** on $\mathbb{C}$ is **zero**, the L^2**-norm** $\|\cdot\|_{L^2(\omega)}$ on any **measurable set** $\omega\subset\mathbb{R}$ of **positive measure** $|\omega|>0$ defines a **norm** on the **finite dimensional vector space** $\mathcal{E}_N$

As a consequence of the **Remez inequality**, this result **holds true** as well in the **multi-dimensional case** when $\omega\subset\mathbb{R}^n$, with $n\geq 1$, is a **measurable subset** of **positive Lebesgue measure** $|\omega|>0$

By **equivalence of norms** in **finite dimension**, for any **measurable set** $\omega\subset\mathbb{R}^n$ of **positive Lebesgue measure** $|\omega|>0$, we have

$$\forall N\in\mathbb{N}, \exists C_N(\omega)>0, \forall f\in\mathcal{E}_N, \quad \|f\|_{L^2(\mathbb{R}^n)}\leq C_N(\omega)\|f\|_{L^2(\omega)}$$

We aim at studying how the **geometrical properties** of the set ω relate to the **growth** of the positive constant $C_N(\omega)>0$ with respect to the **energy level** N

Quantitative spectral estimates for Hermite functions (I)

The following **spectral inequalities** hold :

(i) If ω is a **non-empty open subset** of $\mathbb{R}^n$, then

$$\exists C = C(\omega) > 1, \forall N \in \mathbb{N}, \forall f \in \mathcal{E}_N, \quad \|f\|_{L^2(\mathbb{R}^n)} \leq Ce^{\frac{1}{2}N\ln(N+1)+CN}\|f\|_{L^2(\omega)}$$

(ii) If the **measurable subset** $\omega \subset \mathbb{R}^n$ satisfies the **condition**

$$\liminf_{R\to+\infty} \frac{|\omega \cap B(0,R)|}{|B(0,R)|} > 0$$

then

$$\exists C = C(\omega) > 1, \forall N \in \mathbb{N}, \forall f \in \mathcal{E}_N, \quad \|f\|_{L^2(\mathbb{R}^n)} \leq Ce^{CN}\|f\|_{L^2(\omega)}$$

(iii) If the **measurable subset** $\omega \subset \mathbb{R}^n$ is γ-thick at scale $L > 0$, that is,

$$\forall x \in \mathbb{R}^n, \quad |\omega \cap (x + [0,L]^n)| \geq \gamma L^n$$

then there exist a positive constant $C = C(L, \gamma, n) > 0$ and a universal positive constant $\kappa = \kappa(n) > 0$ such that

$$\forall N \in \mathbb{N}, \forall f \in \mathcal{E}_N, \quad \|f\|_{L^2(\mathbb{R}^n)} \leq C\Big(\frac{\kappa}{\gamma}\Big)^{\kappa L\sqrt{N}}\|f\|_{L^2(\omega)}$$

Beauchard, KPS & Jaming (2018)

Null-controllability of hypoelliptic quadratic equations (I)

Let $q : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \to \mathbb{C}$ be a **complex-valued quadratic form** with a **non negative** real part $\mathrm{Re}\, q \geq 0$, and a **zero singular space** $S = \{0\}$. If ω is a **measurable thick subset** of $\mathbb{R}^n$, then the **parabolic equation**

$$\begin{cases} \partial_t f(t,x) + q^w(x, D_x) f(t,x) = u(t,x) \mathbb{1}_\omega(x), & x \in \mathbb{R}^n, \\ f|_{t=0} = f_0 \in L^2(\mathbb{R}^n), \end{cases}$$

is **null-controllable** from the set ω in **any positive time** $T > 0$

In particular, the **harmonic heat equation**

$$\begin{cases} \partial_t f(t,x) + (-\Delta_x + x^2) f(t,x) = u(t,x) \mathbb{1}_\omega(x), & x \in \mathbb{R}^n, \\ f|_{t=0} = f_0 \in L^2(\mathbb{R}^n), \end{cases}$$

or the **Kramers-Fokker-Planck equation** with a **non degenerate quadratic potential** $V(x) = \frac{1}{2} a x^2$, $a \neq 0$,

$$\begin{cases} \partial_t f(t,v,x) + (-\Delta_v + v^2 + v\partial_x - ax\partial_v) f(t,v,x) = u(t,v,x) \mathbb{1}_\omega(v,x), \\ f|_{t=0} = f_0 \in L^2(\mathbb{R}^2), \qquad (v,x) \in \mathbb{R}^2, \end{cases}$$

are **null-controllable** from any **thick set** ω in **any positive time** $T > 0$

Application to Ornstein-Uhlenbeck equations in L^2_ρ space

Let

$$P = \frac{1}{2}\mathrm{Tr}(Q\nabla_x^2) + \langle Bx, \nabla_x \rangle$$

where Q, $B \in M_n(\mathbb{R})$, with Q **symmetric positive semidefinite**, be a **hypoelliptic Ornstein-Uhlenbeck operator** satisfying the **Kalman rank condition** and the **spectral condition**

$$\mathrm{Rank}[Q^{\frac{1}{2}}, BQ^{\frac{1}{2}}, \dots, B^{n-1}Q^{\frac{1}{2}}] = n, \quad \sigma(B) \subset \mathbb{C}_- = \{z \in \mathbb{C} : \mathrm{Re}\ z < 0\}$$

We consider the operator P acting on the **weighted** L^2-space w.r.t. the **invariant measure** $d\mu(x) = \rho(x)dx$, with **density** w.r.t. Lebesgue measure

$$\rho(x) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det Q_\infty}} e^{-\frac{1}{2}\langle Q_\infty^{-1}x, x\rangle}, \qquad Q_\infty = \int_0^{+\infty} e^{sB} Q e^{sB^T} ds$$

Then, the **Ornstein-Uhlenbeck equation** posed in the L^2_ρ space **weighted** by the **invariant measure**

$$\begin{cases} \partial_t f(t,x) - \frac{1}{2}\mathrm{Tr}[Q\nabla_x^2 f(t,x)] - \langle Bx, \nabla_x f(t,x)\rangle = u(t,x)\mathbb{1}_\omega(x) \\ f|_{t=0} = f_0 \in L^2_\rho \end{cases}$$

is **null-controllable** from the set ω in any time $T > 0$, with a **control function** $u \in L^2((0,T) \times \mathbb{R}^n, dt \otimes \rho(x)dx)$ **supported** in $[0,T] \times \omega$

Is the thickness condition sharp for the null-controllability of the harmonic heat equation ?

Contrary to the **heat equation**, the solutions to the **harmonic heat equation** do enjoy specific **decay properties**

$$\exists C > 1, \exists t_0 > 0, \forall u \in L^2(\mathbb{R}^n), \forall \alpha, \beta \in \mathbb{N}^n, \forall 0 < t \leq t_0,$$

$$\|x^\alpha \partial_x^\beta (e^{-t(D_x^2+x^2)} u)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C^{1+|\alpha|+|\beta|}}{t^{\frac{1}{2}(|\alpha|+|\beta|+2n+s)}} (\alpha!)^{1/2} (\beta!)^{1/2} \|u\|_{L^2(\mathbb{R}^n)}$$

where s is a **fixed integer** verifying $s > \frac{n}{2}$

Question 1 : Do these **decay properties** allow **null-controllability** from subsets with **arbitrary large holes** ? If not, what are the **constraints** on their sizes ?

Question 2 : More generally, if an **evolution equation**

$$\begin{cases} \partial_t f(t,x) + Af(t,x) = u(t,x) \mathbb{1}_\omega(x) \,, & x \in \mathbb{R}^n, \\ f|_{t=0} = f_0 \in L^2(\mathbb{R}^n), \end{cases}$$

enjoys **smoothing properties** for any positive time in some **symmetric Gelfand Shilov spaces** $S_{1/2s}^{1/2s}(\mathbb{R}^n)$ with $1/2 < s \leq 1$ as e.g. the **fractional harmonic oscillator** $A = (D_x^2 + x^2)^s$, how the **index** of **Gelfand Shilov regularity** $1/2s$ relates to the **geometry** of the **control subset** to ensure **null-controllability**

Quantitative spectral estimates for Hermite functions (II)

If the **measurable subset** $\omega \subset \mathbb{R}^n$ satisfies the **condition** :

$\exists 0 < \varepsilon \leq 1$, $\exists 0 < \gamma \leq 1$, $\exists m, R > 0$, $\exists \rho : \mathbb{R}^n \to \mathbb{R}_+$ a $1/2$-**Lipschitz continuous function** verifying

$$\forall x \in \mathbb{R}^n, \quad 0 < m \leq \rho(x) \leq R\langle x\rangle^{1-\varepsilon}$$

such that

$$\forall x \in \mathbb{R}^n, \quad |\omega \cap B(x, \rho(x))| \geq \gamma\, |B(x, \rho(x))|$$

where $B(x, r)$ is a Euclidean ball of $\mathbb{R}^n$ and $|\cdot|$ is the **Lebesgue measure**, then the following **spectral inequality** hold : $\exists \kappa_n(m, R, \gamma, \varepsilon) > 0$, $\exists \tilde{C}_n(\varepsilon, R) > 0$, $\exists \tilde{\kappa}_n > 0$

$$\forall N \geq 1,\ \forall f \in \mathcal{E}_N, \quad \|f\|_{L^2(\mathbb{R}^n)} \leq \kappa_n(m, R, \gamma, \varepsilon)\Big(\frac{\tilde{\kappa}_n}{\gamma}\Big)^{\tilde{C}_n(\varepsilon, R)N^{1-\frac{\varepsilon}{2}}} \|f\|_{L^2(\omega)}$$

Remark. The above result applies for e.g. with $\rho(x) = R\langle x\rangle^{1-\varepsilon}$ with $0 < R \leq \frac{1}{2(1-\varepsilon)}$

Jérémy Martin & KPS (2020)

Null-controllability of hypoelliptic quadratic equations (II)

Let $q : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \to \mathbb{C}$ be a **complex-valued quadratic form** with a **non negative** real part $\mathrm{Re}\, q \geq 0$, and a **zero singular space** $S = \{0\}$. If the **measurable subset** $\omega \subset \mathbb{R}^n$ satisfies the **condition** :

$\exists 0 < \varepsilon \leq 1$, $\exists 0 < \gamma \leq 1$, $\exists m, R > 0$, $\exists \rho : \mathbb{R}^n \to \mathbb{R}_+$ a $1/2$-**Lipschitz continuous function** verifying

$$\forall x \in \mathbb{R}^n, \quad 0 < m \leq \rho(x) \leq R\langle x\rangle^{1-\varepsilon}$$

such that

$$\forall x \in \mathbb{R}^n, \quad |\omega \cap B(x, \rho(x))| \geq \gamma\, |B(x, \rho(x))|$$

then the **parabolic equation**

$$\begin{cases} \partial_t f(t,x) + q^w(x, D_x) f(t,x) = u(t,x) \mathbb{1}_\omega(x), & x \in \mathbb{R}^n, \\ f|_{t=0} = f_0 \in L^2(\mathbb{R}^n), \end{cases}$$

is **null-controllable** from the set ω in **any positive time** $T > 0$

The above **null-controllability result** applies in particular for **harmonic heat equation**, **Kramers-Fokker-Planck equations** with a **non degenerate quadratic potentials** or **hypoelliptic Ornstein-Uhlenbeck equations** in L^2_ρ space

Null-controllability of evolution equations enjoying Gelfand-Shilov smoothing properties

Let $(A, D(A))$ be a **closed operator** on $L^2(\mathbb{R}^n)$ generating a **strongly continuous semigroup** $(e^{-tA})_{t\geq 0}$ on $L^2(\mathbb{R}^n)$ **smoothing** in the **symmetric Gelfand Shilov spaces** $S^{1/2s}_{1/2s}(\mathbb{R}^n)$ with $1/2 < s \leq 1$ such that

$$\exists t_0 > 0, \exists m_1 > 0, \exists m_2 \geq 0, \exists C > 1, \forall \alpha, \beta \in \mathbb{N}^n, \forall u \in L^2(\mathbb{R}^n), \forall 0 < t \leq t_0,$$

$$\|x^\alpha \partial_x^\beta (e^{-tA}u)\| \leq \frac{C^{1+|\alpha|+|\beta|}}{t^{m_1(|\alpha|+|\beta|)+m_2}} (\alpha!)^{\frac{1}{2s}} (\beta!)^{\frac{1}{2s}} \|u\|_{L^2(\mathbb{R}^n)}$$

where $\|\cdot\| = \|\cdot\|_{L^2(\mathbb{R}^n)}$ or $\|\cdot\| = \|\cdot\|_{L^\infty(\mathbb{R}^n)}$. Let $\omega \subset \mathbb{R}^n$ be a **measurable subset** satisfying the **condition** :

$\exists 0 \leq 2-2s < \varepsilon \leq 1$, $\exists 0 < \gamma \leq 1$, $\exists m, R > 0$, $\exists \rho : \mathbb{R}^n \to \mathbb{R}_+$ a $1/2$-**Lipschitz continuous function** verifying $m \leq \rho(x) \leq R\langle x\rangle^{1-\varepsilon}$, $x \in \mathbb{R}^n$, such that

$$\forall x \in \mathbb{R}^n, \quad |\omega \cap B(x, \rho(x))| \geq \gamma\, |B(x, \rho(x))|$$

then the following **evolution equation** is **null-controllable** from the set ω in **any positive time**

$$\begin{cases} \partial_t f(t,x) + Af(t,x) = u(t,x)\mathbb{1}_\omega(x)\,, & x \in \mathbb{R}^n, \\ f|_{t=0} = f_0 \in L^2(\mathbb{R}^n) \end{cases}$$

Thank you for your attention