Quantizations on Nilpotent Lie Groups and Algebras Having Flat Coadjoint Orbits

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joint work with M. Ruzhansky

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Pseudodifferential operators in $\mathrm{G} = \mathbb{R}^n$ and extensions

Definition (Kohn-Nirenberg)

$$[\mathrm{Op}(g)u](x) := \int_{\mathbb{R}^n}\int_{(\mathbb{R}^n)^*} e^{i(x-y)\xi} g(x,\xi)u(y)dxd\xi\,.$$

Problem

Given a "nice" group G , define pseudodifferential operators with configuration space G . Find dual variables belonging to some dual space.

One solution

Phase space $T^*(\mathrm{G})$, pseudodifferential operators on manifolds (Hörmander), using local charts. Very sucesseful, but no full symbolic calculus available (principal symbol), the group structure is obscured.

Interpretations for $\mathrm{G} = \mathbb{R}^n$:

(i) $(\mathbb{R}^n)^* = \widehat{\mathrm{G}}$ the Pontriagin dual, (ii) $(\mathbb{R}^n)^* = \mathfrak{g}^*$ dual of the Lie algebra.

Global quantization with operator valued symbols

G type I unimodular locally compact group with Haar measure m ,
$\widehat{\mathrm{G}} := \mathrm{Irrep}(\mathrm{G})/_\sim$ the unitary dual with Plancherel measure $\widehat{\mathrm{m}}$.
$\{a(x,\xi) : \mathcal{H}_\xi \to \mathcal{H}_\xi \,|\, (x,\xi) \in \mathrm{G}\times\widehat{\mathrm{G}}\}$, where $\pi_\xi : \mathrm{G} \to \mathbb{U}(\mathcal{H}_\xi)$.

Basic formula (M. M. and M. Ruzhansky)

$$\left[\mathrm{Op}_{\mathrm{G}\times\widehat{\mathrm{G}}}(a)u\right](x) = \int_{\mathrm{G}}\int_{\widehat{\mathrm{G}}} \mathrm{Tr}_\xi\left[\pi_\xi(y^{-1}x)a(x,\xi)\right]u(y)d\mathrm{m}(y)d\widehat{m}(\xi)\,.$$

- G Abelian, all irreps are 1-dim. (characters) so $\mathcal{H}_\xi = \mathbb{C}$.
- $\mathrm{G} = \mathbb{R}^n$: $[Op(a)u](x) = \int_{\mathbb{R}^n}\int_{(\mathbb{R}^n)^*} e^{i(x-y)\cdot\xi}a(x,\xi)u(y)dxd\xi$.
- compact Lie (Ruzhansky, Turunen, Wirth, etc), all irreps are finite-dim. so $a(x,\xi)$ is a $d(\xi)^2$-matrix. $\widehat{\mathrm{G}}$ discrete and described by the Peter-Weyl Theorem.
- nilpotent Lie graded (Fisher, Ruzhansky, etc).

The framework

We assume G second countable, and type I (postliminal); this covers Abelian, solvable and compact groups.

The canonical objects in representation theory:
$\mathrm{Rep}(\mathsf{G})\,, \mathrm{Irrep}(\mathsf{G})$ and $\widehat{\mathsf{G}} := \mathrm{Irrep}(\mathsf{G})/_\sim$ (the unitary dual).

The dimension of a representation $\xi : \mathsf{G} \to \mathbb{U}(\mathcal{H}_\xi)$ is $d(\xi) \in \mathbb{N} \cup \{\infty\}$.

The unitary dual $\widehat{\mathsf{G}}$ is endowed with the (standard) Mackey Borel structure and with the Plancherel measure $\widehat{\mathsf{m}}$. If G is Abelian, $\widehat{\mathsf{G}}$ is a locally compact Abelian group and $\widehat{\mathsf{m}}$ is its Haar measure.

On $\Sigma := \mathsf{G} \times \widehat{\mathsf{G}}$, which might not be a locally compact space, we consider the product measure $\mu := \mathsf{m} \otimes \widehat{\mathsf{m}}$.

Mackey's Borel structure

- Any irrep of a second countable group has a separable Hilbert space. Write $\mathrm{Irrep}(\mathsf{G}) = \bigsqcup_{n\in\overline{\mathbb{N}}} \mathrm{Irrep_n}(\mathsf{G})$.
- Define Mackey's Borel structure on $\mathrm{Irrep_n}(\mathsf{G})$.
- $A \subset \mathrm{Irrep}(\mathsf{G})$ Borel iff $A \cap \mathrm{Irrep_n}(\mathsf{G})$ Borel in $\mathrm{Irrep_n}(\mathsf{G})$ for $n \in \overline{\mathbb{N}}$.
- Use the quotient map $\mathrm{Irrep}(\mathsf{G}) \mapsto \widehat{\mathsf{G}}$ to define Borel structure on $\widehat{\mathsf{G}}$.
- For fixed n , let $\mathcal{H}_n$ be an n-dimensional Hilbert space. Via unitary equivalence, all $\xi \in \mathrm{Irrep_n}(\mathsf{G})$ act on $\mathcal{H}_n$.

  For $x \in \mathsf{G}, \xi \in \mathrm{Irrep_n}(\mathsf{G}), u, v \in \mathcal{H}_n$ set $\phi^{\xi}_{u,v}(x) := \langle \xi(x)u, v\rangle_{\mathcal{H}_n}$.

  The smallest Borel structure on $\mathrm{Irrep_n}(\mathsf{G})$ making all the maps $\xi \mapsto \phi^{\xi}_{u,v}(x)$ Borel is The Chosen One.

Type I groups

Definition

The unimodular locally compact second countable group G is type I (postliminal) if for every $\xi \in \mathrm{Irrep}(\mathsf{G})$, the ideal $\mathbb{K}(\mathcal{H}_\xi)$ is contained in the C^*-algebra generated by all the operators

$$r_\xi(h) := \int_{\mathsf{G}} h(x)\xi(x)^* d\mathrm{m}(x)\,, \quad h \in L^1(\mathsf{G})\,.$$

This means that $C^*(\mathsf{G})$ is postliminal (in the sense of C^*-algebras). And equivalent to the fact that the Mackey Borel structure is standard ($\widehat{\mathsf{G}}$ measurably isomorphic to a Borel subset of a complete metric space).

Examples

Abelian, compact, Lie connected semi-simple, Lie connected nilpotent or exponential, Euclidean, Poincaré,....

Counterexamples

Some solvable, many discrete groups, ...

Other results

1. τ-quantizations, including Weyl (symmetric) for certain groups.
2. (Fourier-)Wigner distributions, etc.
3. Connections with crossed-product C^*-algebras associated to topological dynamical systems:
   - The group G acts by left translations L on left-invariant C^*-algebras of functions $\mathcal{A} \subset LUC(\mathrm{G})$.
   - To the algebraic C^*-dynamical system $(\mathcal{A}, L, \mathrm{G})$ one associates a Banach *-algebra $L^1(\mathrm{G}; \mathcal{A})$,
   - then the enveloping C^*-algebra $\mathcal{A} \rtimes_L \mathrm{G}$ whose composition laws are isomorphic to the symbolic calculus and whose Schrödinger representation can be turned isomorphically in the Op calculus.
4. Coherent states, Berezin-Toeplitz-antiWick operators (article).
5. (probably) coorbit (or modulation) spaces.
6. The non-unimodular case (with M. Sandoval), using the Duflo-Moore formalism.
7. Pseudo-differential operators twisted with group 2-cocycles (magnetic type), with H. Bustos.

The nilpotent case

If G is a (simply connected) nilpotent Lie group, two miracles happen:

- $\mathfrak{g} \xrightarrow{\exp} \mathrm{G}$ is a diffeomorphism, with inverse $\mathfrak{g} \xleftarrow{\log} \mathrm{G}$.
- Under exp the Lebesgue measure dX on the Lie algebra $\mathfrak{g}$ is carried into the Haar measure dx of the group G.

So $(\mathrm{G}, \cdot) \cong (\mathfrak{g}, \bullet)$ isomorphic groups for the BCH (polynomial) formula

$$X \bullet Y := \log[\exp(X)\exp(Y)] = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + \frac{1}{12}[Y,[X,Y]] + \dots .$$

A simple-minded quantization would be

$$\left[\mathrm{op}_{\mathrm{G}\times\mathfrak{g}^*}(f)u\right](x)=\int_{\mathrm{G}}\int_{\mathfrak{g}^*}e^{i\langle\log x-\log y|\mathcal{X}\rangle}f\big(x,\mathcal{X}\big)u(y)\,dm(y)d\mathcal{X}.$$

It is just the Kohn-Nirenberg on $\mathfrak{g}\times\mathfrak{g}^*\cong\mathbb{R}^n\times\mathbb{R}^n$ pushed isomorphically to $\mathrm{G}\times\mathfrak{g}^*$ and has nothing to do with the group structure.

Much better ($f:\mathrm{G}\times\mathfrak{g}^*\to\mathbb{C}\,,\,u:\mathrm{G}\to\mathbb{C}$):

$$\left[\mathrm{Op}_{\mathrm{G}\times\mathfrak{g}^*}(f)u\right](x)=\int_{\mathrm{G}}\int_{\mathfrak{g}^*}e^{i\left\langle\log(y^{-1}x)|\mathcal{X}\right\rangle}f\big(x,\mathcal{X}\big)u(y)\,dm(y)\,d\mathcal{X}\,.$$

Composing with exp and log , one gets ($h:\mathfrak{g}\times\mathfrak{g}^*\to\mathbb{C}\,,\,v:\mathfrak{g}\to\mathbb{C}$):

$$\left[\mathrm{Op}_{\mathfrak{g}\times\mathfrak{g}^*}(h)v\right](X)=\int_{\mathfrak{g}}\int_{\mathfrak{g}^*}e^{i\langle(-Y)\bullet X|\mathcal{X}\rangle}h(X,\mathcal{X})\,v(Y)\,dYd\mathcal{X}.$$

The connection

To make the connection between

$$\big[\mathrm{Op}_{\mathsf{G}\times\widehat{\mathsf{G}}}(a)u\big](x)=\int_{\mathsf{G}}\int_{\widehat{\mathsf{G}}}\mathrm{Tr}_\xi\big[\pi_\xi(y^{-1}x)a(x,\xi)\big]u(y)d\mathsf{m}(y)d\widehat{m}(\xi)\,,$$

$$a(x,\xi)\in\mathbb{B}(\mathcal{H}_\xi)\,,\quad (x,\xi)\in\mathsf{G}\times\widehat{\mathsf{G}}$$

and

$$\big[\mathrm{Op}_{\mathsf{G}\times\mathfrak{g}^*}(f)u\big](x)=\int_{\mathsf{G}}\int_{\mathfrak{g}^*}e^{i\langle\log(y^{-1}x)|\mathcal{X}\rangle}f(x,\mathcal{X})u(y)\,d\mathsf{m}(y)\,d\mathcal{X}\,,$$

$$f(x,\mathcal{X})\in\mathbb{C}\,,\quad (x,\mathcal{X})\in\mathsf{G}\times\widehat{\mathsf{G}}$$

one applies in two different directions two different partial Fourier transformations, starting from the integral operator calculus

$$\big[\mathrm{Int}(k)u\big](x):=\int_{\mathsf{G}}k(x,y)u(y)d\mathsf{m}(y)\,.$$

Two unitary Fourier transforms:

- group: $L^2(\mathsf{G}) \xrightarrow{\mathsf{F}_{\mathsf{G},\widehat{\mathsf{G}}}} \mathcal{L}^2(\widehat{\mathsf{G}}) := \int_{\widehat{\mathsf{G}}}^{\oplus} \mathbb{B}^2(\mathcal{H}_\xi) d\widehat{\mathsf{m}}(\xi)\,,$
- vector space: $L^2(\mathsf{G}) \xrightarrow{\circ \exp} L^2(\mathfrak{g}) \xrightarrow{\mathsf{F}_{\mathfrak{g},\mathfrak{g}^*}} L^2(\mathfrak{g}^*)\,.$

Explicitly

$$\big[\mathsf{F}_{\mathsf{G},\widehat{\mathsf{G}}}(u)\big](\xi) = \int_G u(x)\pi_\xi(x)^* d\mathsf{m}(x) \in \mathbb{B}(\mathcal{H}_\xi)\,,$$

$$\Big[\mathsf{F}_{\mathsf{G},\widehat{\mathsf{G}}}^{-1}(\mathfrak{v})\Big](x) = \int_{\widehat{\mathsf{G}}} \mathrm{Tr}_\xi[\pi_\xi(x)\mathfrak{v}(\xi)] d\widehat{\mathsf{m}}(\xi)\,,$$

$$\big[\mathsf{F}_{\mathsf{G},\mathfrak{g}^*}(u)\big](\mathcal{X}) = \int_G e^{-i\langle \log(x)|\mathcal{X}\rangle} u(x) d\mathsf{m}(x)\,,$$

$$\Big[\mathsf{F}_{\mathsf{G},\mathfrak{g}^*}^{-1}(\mathfrak{w})\Big](x) = \int_{\mathfrak{g}^*} e^{i\langle \log(x)|\mathcal{X}\rangle} \mathfrak{w}(\xi) d\mathcal{X}.$$

The composition

Result

Setting $\mathsf{L} := \mathsf{F}_{\mathsf{G},\widehat{\mathsf{G}}} \circ \mathsf{F}^{-1}_{\mathsf{G},\mathfrak{g}^*} : \mathcal{S}(\mathfrak{g}^*) \to \mathcal{S}(\widehat{\mathsf{G}})$
(or at the level of L^2-functions, or at the level of tempered distributions)

$$
\begin{array}{ccc}
L^2(\mathsf{G}) \otimes L^2(\mathsf{G}) & \xrightarrow{\ \mathrm{id}\otimes \mathsf{F}_{\mathsf{G},\widehat{\mathsf{G}}}\ } & L^2(\mathsf{G}) \otimes \mathcal{L}^2(\widehat{\mathsf{G}}) \\
\Big\downarrow {\scriptstyle \mathrm{id}\otimes \mathsf{F}_{\mathsf{G},\mathfrak{g}^*}} & \nearrow {\scriptstyle \mathrm{id}\otimes \mathsf{L}} & \Big\downarrow {\scriptstyle \mathsf{Op}_{\mathsf{G}\times\widehat{\mathsf{G}}}} \\
L^2(\mathsf{G}) \otimes L^2(\mathfrak{g}^*) & \xrightarrow{\ \mathsf{Op}_{\mathsf{G}\times\mathfrak{g}^*}\ } & \mathbb{B}^2[L^2(\mathsf{G})]
\end{array}
$$

$$\mathsf{Op}_{\mathsf{G}\times\mathfrak{g}^*}(A) = \mathsf{Op}_{\mathsf{G}\times\widehat{\mathsf{G}}}[(\mathrm{id}\otimes \mathsf{L})(A)]\,.$$

This allows to transport information from one quantization to the other.
EX: symbol classes $S^m_{\rho,\delta}(\mathsf{G}\times\widehat{\mathsf{G}})$ [Fisher,Ruzhansky] $\longrightarrow$ $\widetilde{S}^m_{\rho,\delta}(\mathsf{G}\times\mathfrak{g}^*)$.
The transformation L is not so explicit and easy to work with.

Kirillov theory

The adjoint action is

$$\mathrm{Ad}\colon \mathsf{G}\times\mathfrak{g}\to\mathfrak{g}\,,\quad \mathrm{Ad}_x(Y) := \frac{d}{dt}\Big|_{t=0}\big[x\exp(tY)x^{-1})\big]$$

and the coadjoint action

$$\mathrm{Ad}^*\colon \mathsf{G}\times\mathfrak{g}^*\to\mathfrak{g}^*,\quad \langle Y\mid \mathrm{Ad}^*(\mathcal{X})\rangle := \langle \mathrm{Ad}_{x^{-1}}(Y)\mid\mathcal{X}\rangle\,.$$

The coadjoint orbit $\Omega\equiv\Omega(\mathcal{X})$ is a closed submanifold with a polynomial structure. There is a Schwartz space $\mathcal{S}(\Omega)$ and a Poisson algebra structure on $\mathfrak{g}^*$ for which the symplectic leaves are the coadjoint orbits.

Result

Homeomorphism $\widehat{\mathsf{G}}\cong\mathfrak{g}^*/\mathrm{Ad}^*,\ \xi\to\Omega_\xi\,.$

The Weyl-Pedersen quantization of coadjoint orbits

Coadjoint orbit $\Omega_\xi \subset \mathfrak{g}^*$, Liouville measure $d\gamma_\xi$.

"Predual" ω_ξ vector subspace of $\mathfrak{g}$ s. t. $\Omega_\xi \ni \mathcal{X} \to \mathcal{X}|_{\omega_\xi} \in \omega_\xi^*$ is a diffeomorphism.

Unitary isomorphism $\mathrm{Ped}_\xi : L^2(\Omega_\xi, \gamma_\xi) \to \mathbb{B}^2(\mathcal{H}_\xi)$. For $\Psi \in \mathcal{S}(\Omega_\xi)$

$$\mathrm{Ped}_\xi(\Psi) := \int_{\omega_\xi}\int_{\Omega_\xi} e^{-i\langle X|\mathcal{X}\rangle}\Psi(\mathcal{X})\xi(\exp X)d\gamma_\xi(\mathcal{X})d\lambda_\xi(X)\,,$$

$$\big[\mathrm{Ped}_\xi^{-1}(S)\big](\mathcal{X}) = \int_{\omega_\xi} e^{i\langle Y|\mathcal{X}\rangle}\mathrm{Tr}_\xi\big[S\,\xi(\exp Y)^*\big]d\lambda_\xi(Y)\,.$$

It generalizes the Weyl calculus on $\mathbb{R}^n \times \mathbb{R}^n \cong$ coadjoint orbit of the $2n+1$ Heisenberg group and satisfies

$$\mathrm{Tr}_\xi\big[\mathrm{Ped}_\xi(\Psi)\big] = \int_{\Omega_\xi} \Psi(\mathcal{X})d\gamma_\xi(\mathcal{X})\,.$$

Flat coadjoint orbits

Definition

The coadjoint orbit $\Omega \equiv \Omega(\mathcal{X}) \equiv \Omega_\xi$ is flat if (equivalent) conditions:

- $\mathfrak{g}_\mathcal{X} := \{Y \in \mathfrak{g} \mid \langle [Y, \cdot] | \mathcal{X} \rangle = 0\} = \mathfrak{z} := \mathrm{Center}(\mathfrak{z})$ ($\supset$ always true).
- $\Omega = \mathcal{U} + \mathfrak{z}^\dagger$ (affine space), where $\mathfrak{z}^\dagger := \{\mathcal{Y} \in \mathfrak{g}^* \mid \mathcal{Y}|_\mathfrak{z} = 0\}$.
- π_ξ is square integrable modulo the center.

The nilpotent group G is admissible if there is a flat coadjoint orbit.

Remark

- If a flat orbit exists, "most of the others" are also flat they are exactly those having maximal dimension.
- An admissible group might not be graded.
- "Many" admissible groups, without being generic.
- All the flat coadjoit orbits have the same predual ω and $\mathfrak{g} = \mathfrak{z} \oplus \omega$.

A nicer form of the composition of the Fourier transforms

Explicit form of L for admissible groups (not true for non-admissible)

- If $B \in \mathcal{S}(\mathfrak{g}^*)$, then $[\mathsf{L}(B)](\xi) = \mathsf{Ped}_\xi\big(B|_{\Omega_\xi}\big)$ and belongs to $\mathcal{S}(\widehat{\mathsf{G}})$.
- Reciprocally, if $b \equiv \big\{b(\xi)\,|\,\xi \in \widehat{\mathsf{G}}\big\} \in \mathcal{S}(\widehat{\mathsf{G}})$, one has

$$\big[\mathsf{L}^{-1}(b)\big](\mathcal{X}) = \int_{\omega_\xi} e^{i\langle Y|\mathcal{X}\rangle}\,\mathrm{Tr}_\xi\big[b(\xi)\xi(\exp Y)^*\big]\,d\lambda_\xi(Y)\,.$$

Corollary

Let ξ be an irreducible representation of the admissible group G , that is square integrable modulo the center, and let

$$\widetilde{\xi}(v) := \int_{\mathsf{G}} v(x)\xi(x)d\mathsf{m}(x)\,, \quad v \in \mathcal{S}(\mathsf{G}),$$

its integrated form acting on the Schwartz space. One has

$$\ker\big(\widetilde{\xi}\big) = \Big\{v \in \mathcal{S}(\mathsf{G})\,\Big|\,\big[\mathsf{F}_{\mathsf{G},\mathfrak{g}^*}(v)\big]|_{\Omega_\xi} = 0\Big\}\,.$$

Main result and other results

Theorem

Suppose that G is admissible and $A \in \mathcal{S}(\mathsf{G} \times \mathfrak{g}^*)$.

- For $(x, \xi) \in \mathsf{G} \times \widehat{\mathsf{G}}$ define $A_{(x,\xi)} := A|_{\{x\} \times \Omega_\xi} \in \mathcal{S}(\Omega_\xi)$
- Set $a(x, \xi) := \mathrm{Ped}_\xi[A_{(x,\xi)}]$ (Weyl-Pedersen calculus assoc. to Ω_ξ) .
- Then you have $a = (\mathrm{id} \otimes \mathsf{L})A$, i. e. $\mathrm{Op}_{\mathsf{G} \times \mathfrak{g}^*}(A) = \mathrm{Op}_{\mathsf{G} \times \widehat{\mathsf{G}}}(a)$.

1. Using parametrizations of the unitary dual (Pukansky, Moore+Wolf) one could give more concrete versions of the pseudodifferential calculus $\mathrm{Op}_{\mathsf{G} \times \widehat{\mathsf{G}}}$ (extended to classes of solvable groups in [M. M. and M. Sandoval, Monatsh. Math.], using Curry's parametrization) . There is a version of the Theorem in this setting too.
2. This simplifies for Lie algebras with 1-dimensional center. For the Heisenberg group for example $\xi \equiv \lambda \in \mathbb{R} \setminus \{0\}$ and the connection between the two quantizations involve packing together all the $\hbar$-dependent Weyl calculi.
3. Some other explicit examples worked out.

Supplementary stuff

In the next slides I will add some extra information about the global pseudodifferential calculus with operator-valued symbols $\mathrm{Op} \equiv \mathrm{Op}_{\mathrm{G}\times\mathfrak{g}^*}$ that do not refer to nilpotent groups but which might be interesting.

This is joint work with M. Ruzhansky and M. Sandoval.

τ-pseudodifferential operators

Let $\tau : \mathsf{G} \to \mathsf{G}$ measurable, as $\tau(x) := e$ or $\tau(x) := x$ for instance.

Applying the "Schrödinger machine" to the τ-versions of the crossed products one computes $\mathsf{Op}^\tau := (r \rtimes^\tau U) \circ (\mathsf{F} \otimes 1)$ and get

$$[\mathsf{Op}^\tau(f)v](x) = \int_{\mathsf{G}} \int_{\widehat{\mathsf{G}}} \mathrm{Tr}\big[\xi(xy^{-1}) f(\xi, \tau(yx^{-1})x)\big] \, v(y) d\mathsf{m}(y) d\widehat{\mathsf{m}}(\xi) \,.$$

Notice that, when G is commutative, $\tau(yx^{-1})x = (1-\tau)x + \tau y$.

If $\mathsf{G} = \mathbb{R}^n$ one can take $\tau := \tau \mathrm{id}$ with $\tau \in [0,1]$; then $\tau = 1/2$ is very important. But there are other endomorphisms (projections for instance).

Multiplication and convolution operators

Multiplication operator with (reasonable, continuous) $a : \mathsf{G} \to \mathsf{G}$:

$$\mathrm{Mult}(a) : L^2(\mathsf{G}) \to L^2(\mathsf{G}) \,, \quad \mathrm{Mult}(a)v := av \,.$$

Left convolution operator with $b \in L^1(\mathsf{G})$:

$$\mathrm{Conv}_L(b) : L^2(\mathsf{G}) \to L^2(\mathsf{G}) \,, \quad \mathrm{Conv}_L(b)v := b * v \,.$$

If $b = F^{-1}(\beta)$ and $f(\xi, x) := (\beta \otimes a)(\xi, x) := a(x)\beta(\xi)$ then

- For $\tau(x) = \mathsf{e}$, we get (left quantization)

$$\mathrm{Op}(\beta \otimes a) = \mathrm{Mult}(a)\mathrm{Conv}_L(b)$$

- For $\tau(x) = x$, we get (right quantization)

$$\mathrm{Op}^{\mathrm{id}}(\beta \otimes a) = \mathrm{Conv}_L(b)\mathrm{Mult}(a) \,.$$

Symmetric quantizations

One has $\mathrm{Op}^{\tau}(f)^* = \mathrm{Op}^{\tilde{\tau}}(f^{\star})$, where $\tilde{\tau}(x) := \tau(x^{-1})x$ and

$$f^{\star}(\xi, x) := f(\xi, x)^* \quad (in\ \mathbb{B}(\mathcal{H}_{\xi}))$$

Definition

The τ-quantization $f \to \mathrm{Op}^{\tau}(f)$ is called symmetric if $\tilde{\tau} = \tau$,
i.e. "real" symbols are sent into self-adjoint operators.

Examples

For $\mathrm{G} = \mathbb{R}^n$, if one assumes $\tau(-x) = \tau(x)$ for every $x \in \mathbb{R}^n$, this means $\tau(x) = x/2$ and one gets the Weyl quantization.

For $\mathrm{G} = \mathbb{Z}^n$ this is not possible!

Exponential Lie groups (ex: nilpotent) admit a symmetric quantization.

The property is stable under direct products and central extensions.

The key objects $(\widehat{\mathsf{G}}, \widehat{\mathsf{m}})$ are in general complicated and abstract.

Main topic in Harmonic Analysis: Determine them and/or give them a human face (reinterpretations, parametrizations, etc)!

Kirillov Theory: Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{g}^\sharp$ its dual. Describe $\widehat{\mathsf{G}}$ through the orbit space $\mathfrak{g}^\sharp/_{\mathsf{G}}$ for coadjoint action of G on $\mathfrak{g}^\sharp$:

$$\mathrm{Ad}_x(Y) = \frac{d}{dt}\Big|_{t=0} x \exp(tY)x^{-1}\,, \quad \mathrm{Ad}_x^\sharp(\theta) = \theta \circ \mathrm{Ad}_x^{-1}.$$

Kirillov map (not very good in general): $\kappa : \mathfrak{g}^\sharp/_{\mathsf{G}} \to \widehat{\mathsf{G}}$.

If G exponential, then κ is a homeomorphism ([Leptin+Ludwig]; difficult).

However in $\underset{Plancherel}{\widehat{\mathsf{G}}} \xrightarrow{\kappa^{-1}} \mathfrak{g}^\sharp/_{\mathsf{G}} \xleftarrow{q} \underset{Lebesgue}{\mathfrak{g}^\sharp}$

the measures $\kappa^{-1}(Plancherel)$ and $q(Lebesgue)$ are equivalent, but not equal in general. (They are equal in the nilpotent case!)

The orbital form of the global quantization

Measures μ on $\mathfrak{g}$ that are $\mathrm{Ad}^\sharp$-invariant have the form $d\mu_\Psi(\theta) = \Psi(\theta)d\theta$, where $\Psi : \mathfrak{g}^\sharp \to \mathbb{R}_+$ is Δ^{-1}-semi-invariant:

$$\Psi\big[\mathrm{Ad}^\sharp_x(\theta)\big] = \Delta(x)^{-1}\Psi(\theta)\,.$$

They exist and can be taken rational, cf. [Duflo+Raïs].

One has unique decompositions $\mu_\Psi = \int_{\mathfrak{g}^\sharp/\mathsf{G}} \omega_{\mathcal{O}} d\nu_\Psi(\mathcal{O})$, meaning

$$\int_{\mathfrak{g}^\sharp} h(\theta)\Psi(\theta)d\theta = \int_{\mathfrak{g}^\sharp/\mathsf{G}} \Big[\int_{\mathcal{O}} h(\theta)d\omega_{\mathcal{O}}(\theta)\Big] d\nu_\Psi(\mathcal{O})\,, \quad h \in C_{\rm c}^\infty(\mathfrak{g}^\sharp)\,.$$

Here $\omega_{\mathcal{O}}$ is the Liouville measure on the coadjoint orbit $\mathcal{O}$ (a symplectic manifold) and ν_Ψ is a "basic measure" on $\mathfrak{g}^\sharp/\mathsf{G}$.

Then an orbital Plancherel Theorem is used in the orbital quantization:

$$\mathsf{Op}_{\rm orb} : L^2(\mathsf{G}) \otimes \mathcal{B}^2(\mathfrak{g}^\sharp/\mathsf{G}, \nu_\Psi) \to \mathbb{B}^2\big[L^2(\mathsf{G})\big]\,,$$

$$\big[\mathsf{Op}_{\rm orb}(B)u\big](x) = \int_{\mathsf{G}}\int_{\mathfrak{g}^\sharp/\mathsf{G}} \mathrm{Tr}_\xi\big[B(x,\mathcal{O})\,\mathrm{D}^{1/2}_{\Psi,\mathcal{O}}\pi_{\mathcal{O}}(yx^{-1})\big]\Delta(y)^{-\frac{1}{2}}u(y)dm(y)d\nu_\Psi(\mathcal{O})\,.$$

Parametrizations

One still needs to understand $\mathfrak{g}^\sharp/_{\mathsf{G}}$ and ν_Ψ better.

Example

If G is a connected Lie group, then [Ingrid, Daniel]

- the family of open coadjoint orbits is finite,
- its union is a Zariski open subset of $\mathfrak{g}^\sharp$.

So, if there is an open coadjoint orbit, $\mathfrak{g}^\sharp/_{\mathsf{G}} = \text{finite set} \sqcup \text{"small" set}$ and

$$\big[\mathsf{Op}_{\rm orb}(B)u\big](x) = \int_{\mathsf{G}} \sum_{k=1}^{N} {\rm Tr}\big[B_k(x)\,{\rm D}_k^{1/2}\pi_k(yx^{-1})\big]\,\Delta(y)^{-\frac{1}{2}}u(y)dm(y)\,.$$

For the $ax+b$ group $\mathbb{R}\rtimes\mathbb{R}_+$ one has $\text{finite set} = \pm$ and ${\rm D}_\pm, \pi_\pm$ are known.

Similarly, other semidirect products.

Currey's parametrizations for exponential $\Leftarrow$ completely solvable $\Leftarrow$ Nilpotent

Theorem

There is an $\mathrm{Ad}^\sharp$-invariant stratification $\mathfrak{g}^\sharp \underset{\sim}{\supset} \Omega = \bigsqcup_{\epsilon\in E} \Omega_\epsilon$.

For each ϵ, homeomorphism $\Omega_\epsilon/_{\mathsf{G}} \cong \Lambda_\epsilon = \mathrm{Zariski\ open\ subset\ of\ } \mathbb{R}^{\mathrm{n}_\epsilon}$.

The restriction $\nu_{\Psi,\epsilon}$ of ν_Ψ to $\Omega_\epsilon/_{\mathsf{G}}$ is transformed into $\gamma_{\psi,\epsilon}(\lambda)d\lambda$ with $\gamma_{\Psi,\epsilon}$ computable.

One gets the concrete form of the global quantization

$$\big[\mathrm{Op}_{\rm con}(\mathcal{B})u\big](x) =$$

$$= \sum_{\epsilon\in E} \int_{\mathsf{G}} \int_{\Lambda_\epsilon} \mathrm{Tr}\big[\mathcal{B}(x,\lambda)\,\mathrm{D}_{\Psi,\lambda}^{1/2}\,\pi_\lambda(yx^{-1})\big]\Delta(y)^{-\frac{1}{2}}u(y)\,\gamma_{\Psi,\epsilon}(\lambda)d\lambda\,dm(y)\,.$$

Very explicit parametrizations and densities in certain cases (Bianchi classification, others).