R-actions and invariant differential operators

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Overview

A closed manifold $\mathcal{M}$ together with a nowhere vanishing real vector field $\mathcal{T}$ that is a Killing vector field for some Riemannian metric behaves in many ways like a circle bundle over a compact base.
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To illustrate the point, I will discuss a theorem concerning the action of such a vector field on the kernel of an invariant differential operator

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P : C^\infty(M; E) \to C^\infty(M; E), \quad [\mathcal{L}_\mathcal{T}, P] = 0,
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acting on sections of a Hermitian vector bundle,
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$$P : C^\infty(M; E) \to C^\infty(M; E), \quad [\mathcal{L}_\mathcal{T}, P] = 0, \quad PP^* = P^*P$$

acting on sections of a Hermitian vector bundle, assumed to be normal with respect to the natural Hilbert space structure.

On $\ker P \subset L^2$, $\mathcal{L}_\mathcal{T}$ will act as a Fredholm selfadjoint operator with compact parameter (assuming some sort of ellipticity).

As an application I will discuss how with certain hypoellipticity condition one gets a result resembling Kodaira's vanishing theorem.

At the end I will sketch the basic ideas of the proofs.
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At the end I will sketch the basic ideas of the proofs.
Set-up

\( M \) is a closed manifold, compact no boundary (and connected)
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\(\mathcal{T}\) is a nowhere vanishing real vector field preserving some Riemannian metric \(g\)
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$M$ is a closed manifold, $\mathcal{L}_T g = 0$. We only care that such $g$ exists.

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$\alpha_t$ is the flow of $T$, $m$ is an invariant smooth density
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\[ T \text{ is a nowhere vanishing real vector field preserving some Riemannian metric } g \]

\[ a_t \text{ is the flow of } T, \ m \text{ is an invariant smooth density} \]

\[ E \to M \text{ is a Hermitian vector bundle, metric } h \]
Set-up

\( M \) is a closed manifold, \( \mathcal{L}_Tg = 0. \)  We only care that such \( g \) exists

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\( \mathcal{U}_t : E \to E \) is a unitary bundle homomorphism covering \( \alpha_t \)
Set-up

\[ M \text{ is a closed manifold,} \quad \mathcal{L}_T g = 0. \text{ We only care that such } g \text{ exists} \]

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Lie derivative:

\[
\begin{align*}
E & \xrightarrow{\mathcal{A}_t} E \\
M & \xrightarrow{a_t} M
\end{align*}
\]
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Lie derivative:

- \( \phi \) section of \( E \), \( p \in M \).
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Lie derivative:

$\phi$ section of $E$, $p \in M$. $a_t(p)$

\[ E \xrightarrow{\mathcal{A}_t} E \]

\[ M \xrightarrow{a_t} M \]
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\[ \phi \text{ section of } E, \ p \in M. \quad \phi(a_t(p)) \]
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Lie derivative:

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Lie derivative:

\[ \phi \text{ section of } E, \ p \in M. \ t \mapsto \mathcal{A}_{-t}(\phi(a_t(p))) \text{ is a curve in } E_p. \]
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Lie derivative:

$\phi$ section of $E$, $p \in M$. $t \mapsto \mathcal{A}_t(\phi(a_t(p)))$ is a curve in $E_p$, $\frac{d}{dt} \bigg|_{t=0} \mathcal{A}_t(\Phi(a_t(p)))$.
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Lie derivative:

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\phi \text{ section of } E, \ p \in M. \ t \mapsto \mathcal{A}_{-t}(\phi(a_t(p))) \text{ is a curve in } E_p,
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\[
\mathcal{L}_T(\phi)(p) = \frac{d}{dt} \bigg|_{t=0} \mathcal{A}_{-t}(\Phi(a_t(p))).
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Set-up

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Lie derivative:

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$P$ is a differential operator $C^\infty(M; E) \to C^\infty(M; E)$, of order $m$
Set-up

$M$ is a closed manifold, $\mathcal{L}_T g = 0$. We only care that such $g$ exists.

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\mathcal{L}_T(\phi)(p) = \left. \frac{d}{dt} \right|_{t=0} \mathcal{U}_{-t}(\Phi(a_t(p))).
\]

$P$ is a differential operator $C^\infty(M; E) \to C^\infty(M; E)$. We ask of order $m$

$[\mathcal{L}_T, P] = 0$, $[P, P^*] = 0$
Set-up

$M$ is a closed manifold, $\mathcal{L}_\mathcal{T} g = 0$. We only care that such $g$ exists.

$\mathcal{T}$ is a nowhere vanishing real vector field preserving some Riemannian metric $g$.

$a_t$ is the flow of $\mathcal{T}$, $m$ is an invariant smooth density.

$E \to M$ is a Hermitian vector bundle, metric $h$

$\mathcal{A}_t : E \to E$ is a unitary bundle homomorphism covering $a_t$

Lie derivative:

$\phi$ section of $E$, $p \in M$. $t \mapsto \mathcal{A}_{-t}(\phi(a_t(p)))$ is a curve in $E_p$,

$L_\mathcal{T}(\phi)(p) = \frac{d}{dt} \bigg|_{t=0} \mathcal{A}_{-t}(\Phi(a_t(p)))$.

$P$ is a differential operator $C^\infty(M; E) \to C^\infty(M; E)$. We ask of order $m$

$[\mathcal{L}_\mathcal{T}, P] = 0$, $[P, P^*] = 0$

$\sigma(P) + \sigma(-i\mathcal{L}_\mathcal{T})^m - \lambda I$ is invertible if $\lambda \in \Lambda$. 

$E \xrightarrow{\mathcal{A}_t} E$

$M \xrightarrow{a_t} M$
**Set-up**

\( M \) is a closed manifold, \( T \) is a nowhere vanishing real vector field preserving some Riemannian metric \( g \). \( a_t \) is the flow of \( T \), \( m \) is an invariant smooth density. 

\( E \to M \) is a Hermitian vector bundle, metric \( h \). \( \mathcal{A}_t : E \to E \) is a unitary bundle homomorphism covering \( a_t \).

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[\mathcal{L}_T, P] = 0, \ [P, P^*] = 0 \\
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$M$ is a closed manifold, $\mathcal{L}_g = 0$. We only care that such $g$ exists.

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Lie derivative:

- $\phi$ section of $E$, $p \in M$. $t \mapsto \mathcal{A}_{-t}(\phi(a_t(p)))$ is a curve in $E_p$,

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- $[\mathcal{L}_T, P] = 0$, $[P, P^*] = 0$

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$E \xrightarrow{\mathcal{A}_t} E$

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\( M \) is a closed manifold, \( \mathcal{L}_T g = 0 \). We only care that such \( g \) exists

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\( P \) is a differential operator \( C^\infty(M; E) \to C^\infty(M; E) \). We ask

of order \( m \)

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[\mathcal{L}_T, P] = 0, \ [P, P^*] = 0
\]

(specifically \( P + (-i\mathcal{L}_T)^m \) is elliptic)

\[
\sigma(P) + \sigma(-i\mathcal{L}_T)^m - \lambda I \text{ is invertible if } \lambda \in \Lambda.
\]

Let \( \mathscr{D} = \{ \phi \in H : \mathcal{L}_T \phi \in H \} \).

\( H = \ker P \cap L^2 \) is a Hilbert space on its own.
Set-up, Theorem

$M$ is a closed manifold,
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$a_t$ is the flow of $\mathcal{T}$, $\mathfrak{m}$ is an invariant smooth density
$E \rightarrow M$ is a Hermitian vector bundle, metric $h$
$\mathcal{A}_t : E \rightarrow E$ is a unitary bundle homomorphism covering $a_t$

Lie derivative:

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\phi \text{ section of } E, \; p \in M. \; t \mapsto \mathcal{A}_{-t}(\phi(a_t(p))) \text{ is a curve in } E_p, \\
\mathcal{L}_\mathcal{T}(\phi)(p) = \frac{d}{dt} \bigg|_{t=0} \mathcal{A}_{-t}(\Phi(a_t(p))).
\]

$P$ is a differential operator $\mathcal{C}^\infty(M; E) \rightarrow \mathcal{C}^\infty(M; E)$. We ask

of order $m$

$[\mathcal{L}_\mathcal{T}, P] = 0, \; [P, P^*] = 0$

$\sigma(P) + \sigma(-i\mathcal{L}_\mathcal{T})^m - \lambda I$ is invertible if $\lambda \in \Lambda$.

Let $\mathcal{D} = \{ \phi \in H : \mathcal{L}_\mathcal{T}\phi \in H \}$.

\[-i\mathcal{L}_\mathcal{T} \big|_{\mathcal{D}} : \mathcal{D} \subset H \rightarrow H\]

$H = \ker P \cap L^2$ is a Hilbert space on its own.
Set-up, Theorem

\( M \) is a closed manifold, \( \mathcal{L}_T g = 0 \). We only care that such \( g \) exists.

\( T \) is a nowhere vanishing real vector field preserving some Riemannian metric \( g \).
\( a_t \) is the flow of \( T \), \( \mathfrak{m} \) is an invariant smooth density.

\( E \to M \) is a Hermitian vector bundle, metric \( h \).
\( \mathcal{U}_t : E \to E \) is a unitary bundle homomorphism covering \( a_t \).

\( \mathfrak{A}_t : E \to E \) is a unitary bundle homomorphism covering \( a_t \).

\[ \mathcal{L}_T (\phi) (p) = \frac{d}{dt} \bigg|_{t=0} \mathcal{A}_{-t} (\Phi (a_t (p))). \]

\( P \) is a differential operator \( C^\infty (M; E) \to C^\infty (M; E) \). We ask of order \( m \)

\[ [\mathcal{L}_T, P] = 0, \quad [P, P^*] = 0 \]

so \( P + (-i\mathcal{L}_T)^m \) is elliptic.

\( \sigma (P) + \sigma (-i\mathcal{L}_T)^m - \lambda I \) is invertible if \( \lambda \in \Lambda \).

\( \mathcal{D} = \{ \phi \in H : \mathcal{L}_T \phi \in H \} \).

\[ -i\mathcal{L}_T \big|_{\mathcal{D}} : \mathcal{D} \subset H \to H \]
is selfadjoint with compact resolvent, in particular Fredholm.

\[ H = \ker P \cap L^2 \] is a Hilbert space on its own.
Circle bundle?

$M$ is a closed manifold, $\mathcal{L}_T g = 0$. We only care that such $g$ exists.

$T$ is a nowhere vanishing real vector field preserving some Riemannian metric $g$.

$\alpha_t$ is the flow of $T$, $m$ is an invariant smooth density.

$E \to M$ is a Hermitian vector bundle, metric $h$.

$\mathfrak{A}_t : E \to E$ is a unitary bundle homomorphism covering $\alpha_t$.

Lie derivative:

$\phi$ section of $E$, $p \in M$. $t \mapsto \mathfrak{A}_{-t}(\phi(\alpha_t(p)))$ is a curve in $E_p$,

$$\mathcal{L}_T(\phi)(p) = \left. \frac{d}{dt} \right|_{t=0} \mathfrak{A}_{-t}(\Phi(\alpha_t(p))).$$

$P$ is a differential operator $C^\infty(M; E) \to C^\infty(M; E)$. We ask

$[\mathcal{L}_T, P] = 0$, $[P, P^*] = 0$ (so $P + (-i\mathcal{L}_T)^m$ is elliptic).

$\sigma(P) + \sigma((-i\mathcal{L}_T)^m) - \lambda I$ is invertible if $\lambda \in \Lambda$.

Let $\mathcal{D} = \{ \phi \in H : \mathcal{L}_T \phi \in H \}$.

$-i\mathcal{L}_T|_\mathcal{D} : \mathcal{D} \subset H \to H$

is selfadjoint with compact resolvent, in particular Fredholm.
Circle bundle?

For \( p \in M \): \( O_p = \) orbit of \( p \), say \( p \sim p' \) iff \( p' \in \overline{O}_p \).

\[ M \text{ is a closed manifold, } \mathcal{L}_T g = 0. \text{ We only care that such } g \text{ exists} \]

\( T \) is a nowhere vanishing real vector field preserving some Riemannian metric \( g \)

\( a_t \) is the flow of \( T \), \( m \) is an invariant smooth density

\( E \to M \) is a Hermitian vector bundle, metric \( h \)

\( \mathfrak{A}_t : E \to E \) is a unitary bundle homomorphism covering \( a_t \)

Lie derivative:

\[ \phi \text{ section of } E, \ p \in M. \ t \mapsto \mathfrak{A}_{-t}(\phi(a_t(p))) \text{ is a curve in } E_p, \]

\[ \mathcal{L}_{T}(\phi)(p) = \left. \frac{d}{dt} \right|_{t=0} \mathfrak{A}_{-t}(\Phi(a_t(p))). \]

\( P \) is a differential operator \( C^\infty(M; E) \to C^\infty(M; E) \). We ask

\[ [\mathcal{L}_{T}, P] = 0, \ [P, P^*] = 0 \]

so \( P + (-i\mathcal{L}_{T})^m \) is elliptic

\( \sigma(P) + \sigma(-i\mathcal{L}_{T})^m - \lambda I \) is invertible if \( \lambda \in \Lambda. \)

Let \( \mathcal{D} = \{ \phi \in H : \mathcal{L}_{T}\phi \in H \}. \)

\[ -i\mathcal{L}_{T} |_{\mathcal{D}} : \mathcal{D} \subset H \to H \]

is selfadjoint with compact resolvent, in particular Fredholm.
Circle bundle?

For \( p \in M: \mathcal{O}_p = \text{orbit of } p, \)

say \( p \sim p' \) iff \( p' \in \overline{\mathcal{O}_p}, \)

let \( \phi: M \to B := M/\sim. \)

The fibers of \( \phi \) are tori.

\[ M \] is a closed manifold, \( \mathcal{L}_T g = 0. \) We only care that such \( g \) exists

\( T \) is a nowhere vanishing real vector field preserving some Riemannian metric \( g \)

\( \alpha_t \) is the flow of \( T, \) \( m \) is an invariant smooth density \( E \to M \) is a Hermitian vector bundle, metric \( h \)

\( \mathfrak{A}_t: E \to E \) is a unitary bundle homomorphism covering \( \alpha_t \)

Lie derivative:

\( \phi \) section of \( E, \) \( p \in M. \) \( t \mapsto \mathfrak{A}_{-t}(\Phi(\alpha_t(p))) \) is a curve in \( E_p, \)

\[ \mathcal{L}_T(\phi)(p) = \frac{d}{dt} \bigg|_{t=0} \mathfrak{A}_{-t}(\Phi(\alpha_t(p))). \]

\( P \) is a differential operator \( C^\infty(M; E) \to C^\infty(M; E). \) We ask

\[ [\mathcal{L}_T, P] = 0, [P, P^*] = 0 \]

(\( so \) \( P + (-i\mathcal{L}_T)^m \) is elliptic)

\[ \sigma(P) + \sigma(-i\mathcal{L}_T)^m - \lambda I \text{ is invertible if } \lambda \in \Lambda. \]

Let \( \mathcal{D} = \{ \phi \in H : \mathcal{L}_T \phi \in H \}. \)

\[ -i\mathcal{L}_T |_{\mathcal{D}} : \mathcal{D} \subset H \to H \]

is selfadjoint with compact resolvent, in particular Fredholm.
Circle bundle?

For \( p \in M \): \( O_p = \text{orbit of } p \), say \( p \sim p' \) iff \( p' \in \overline{O}_p \), let \( \varphi : M \to B := M/\sim \). The fibers of \( \varphi \) are tori.

Reason: The closure of \( \alpha_t \) in \( \text{Iso}(M) \) is isomorphic to a torus \( \mathbb{T} \).

\( M \) is a closed manifold, \( \mathcal{L}_T g = 0 \). We only care that such \( g \) exists.

\( T \) is a nowhere vanishing real vector field preserving some Riemannian metric \( g \)
\( \alpha_t \) is the flow of \( T \), \( m \) is an invariant smooth density
\( E \to M \) is a Hermitian vector bundle, metric \( h \)
\( \mathfrak{A}_t : E \to E \) is a unitary bundle homomorphism covering \( \alpha_t \)

Lie derivative:

- \( \phi \) section of \( E \), \( p \in M \). \( t \mapsto \mathfrak{A}_{-t}(\phi(\alpha_t(p))) \) is a curve in \( E_p \),
- \( \mathcal{L}_T(\phi)(p) = \frac{d}{dt} \big|_{t=0} \mathfrak{A}_{-t}(\Phi(\alpha_t(p))) \).

\( P \) is a differential operator \( C^\infty(M; E) \to C^\infty(M; E) \). We ask of order \( m \)

\( [\mathcal{L}_T, P] = 0, \ [P, P^*] = 0 \) (so \( P + (-i\mathcal{L}_T)^m \) is elliptic)
\( \sigma(P) + \sigma(-i\mathcal{L}_T)^m - \lambda I \) is invertible if \( \lambda \in \Lambda \).

Let \( \mathcal{D} = \{ \phi \in H : \mathcal{L}_T \phi \in H \} \).
- \( -i\mathcal{L}_T |_\mathcal{D} : \mathcal{D} \subset H \to H \)
is selfadjoint with compact resolvent, in particular Fredholm.
Circle bundle?

For \( p \in M \): \( \mathcal{O}_p = \text{orbit of } p \), say \( p \sim p' \) iff \( p' \in \mathcal{O}_p \), let \( \varphi : M \to B := M/\sim \).

The fibers of \( \varphi \) are tori.

Reason:
The closure of \( \alpha_t \) in \( \text{Iso}(M) \) is isomorphic to a torus \( \mathbb{T} \).

There are open dense sets \( M^{\text{reg}} \subset M, B^{\text{reg}} \subset B \) such that \( \pi : M^{\text{reg}} \to B^{\text{reg}} \) is a principal torus bundle.

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\( E \to M \) is a Hermitian vector bundle, metric \( h \).
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Lie derivative:
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\phi \text{ section of } E, \ p \in M. \ t \mapsto \mathcal{A}_{-t}(\phi(\alpha_t(p))) \text{ is a curve in } E_p,
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\mathcal{L}_T(\phi)(p) = \frac{d}{dt} \bigg|_{t=0} \mathcal{A}_{-t}(\Phi(\alpha_t(p))).
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\( P \) is a differential operator \( C^\infty(M; E) \to C^\infty(M; E) \). We ask \( P \) of order \( m \)
\[ [\mathcal{L}_T, P] = 0, [P, P^*] = 0 \]
(\( \text{so } P + (-i\mathcal{L}_T)^m \text{ is elliptic} \))
\[ \sigma(P) + \sigma(-i\mathcal{L}_T)^m - \lambda I \text{ is invertible if } \lambda \in \Lambda. \]

Let \( \mathcal{D} = \{ \phi \in H : \mathcal{L}_T \phi \in H \} \).
\[ -i\mathcal{L}_T \big| _\mathcal{D} : \mathcal{D} \subset H \to H \]
is selfadjoint with compact resolvent, in particular Fredholm.
Circle bundle?

For \( p \in M \): \( O_p = \) orbit of \( p \), say \( p \sim p' \) iff \( p' \in \overline{O}_p \), let \( \varphi : M \to B := M/\sim \). The fibers of \( \varphi \) are tori.

Reason: The closure of \( \alpha_t \) in \( \text{Iso}(M) \) is isomorphic to a torus \( \mathbb{T} \).

There are open dense sets \( M^{\text{reg}} \subset M \), \( B^{\text{reg}} \subset B \) such that \( \pi : M^{\text{reg}} \to B^{\text{reg}} \) is a principal torus bundle.

Example. Let \( B \) be a compact complex manifold, \( \varphi : M \to B \) the circle bundle of a holomorphic line bundle \( L \to B \),

\[ M = \{ \eta \in L : |\eta|^2 = 1 \} \text{ for some Hermitian metric} \]

\[ T \text{ is a nowhere vanishing real vector field preserving some Riemannian metric } g \]
\[ \alpha_t \text{ is the flow of } T, m \text{ is an invariant smooth density} \]
\[ E \to M \text{ is a Hermitian vector bundle, metric } h \]
\[ \mathcal{A}_t : E \to E \text{ is a unitary bundle homomorphism covering } \alpha_t \]

Lie derivative:
\[ \phi \text{ section of } E, p \in M, t \mapsto \mathcal{A}_{-t}(\phi(\alpha_t(p))) \text{ is a curve in } E_p, \]
\[ \mathcal{L}_T(\phi)(p) = \left. \frac{d}{dt} \right|_{t=0} \mathcal{A}_{-t}(\Phi(\alpha_t(p))). \]

\[ P \text{ is a differential operator } C^\infty(M;E) \to C^\infty(M;E). \text{ We ask} \]
\[ [\mathcal{L}_T, P] = 0, [P, P^*] = 0 \text{ (so } P + (-i\mathcal{L}_T)^m \text{ is elliptic)} \]
\[ \sigma(P) + \sigma(-i\mathcal{L}_T)^m - \lambda I \text{ is invertible if } \lambda \in \Lambda. \]

Let \( \mathcal{D} = \{ \phi \in H : \mathcal{L}_T \phi \in H \}. \]
\[ -i\mathcal{L}_T|_{\mathcal{D}} : \mathcal{D} \subset H \to H \]

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Circle bundle?

For \( p \in M \): \( O_p = \text{orbit of } p \), say \( p \sim p' \) iff \( p' \in \overline{O}_p \).

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\( M \) is a closed manifold, \( \mathcal{L}_T g = 0 \). We only care that such \( g \) exists.

\( T \) is a nowhere vanishing real vector field preserving some Riemannian metric \( g \)
\( \alpha_t \) is the flow of \( T \), \( m \) is an invariant smooth density
\( E \to M \) is a Hermitian vector bundle, metric \( h \)
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Lie derivative:
\( \phi \) section of \( E \), \( p \in M. t \mapsto \mathcal{A}_{-t}(\phi(\alpha_t(p))) \) is a curve in \( E_p \),
\[
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\]

\( P \) is a differential operator \( C^\infty(M; E) \to C^\infty(M; E). \) We ask
\[
[\mathcal{L}_T, P] = 0, [P, P^*] = 0 \quad \text{(so } P + (-i\mathcal{L}_T)^m \text{ is elliptic)}
\]
\[
\sigma(P) + \sigma(-i\mathcal{L}_T)^m - \lambda I \text{ is invertible if } \lambda \in \Lambda.
\]

Let \( \mathcal{D} = \{ \phi \in H : \mathcal{L}_T \phi \in H \} \).

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Circle bundle?

For $p \in M$: $\mathcal{O}_p = \text{orbit of } p$, say $p \sim p'$ iff $p' \in \overline{\mathcal{O}_p}$, let $\varphi : M \to B := M/\sim$.

The fibers of $\varphi$ are tori.

Reason:

The closure of $\alpha_t$ in $\text{Iso}(M)$ is isomorphic to a torus $\mathbb{T}$.

There are open dense sets $M^{\text{reg}} \subset M$, $B^{\text{reg}} \subset B$ such that $\pi : M^{\text{reg}} \to B^{\text{reg}}$ is a principal torus bundle.

Example. Let $B$ be a compact complex manifold, $\varphi : M \to B$ the circle bundle of a holomorphic line bundle $L \to B$, $T$ the generator of $t \mapsto e^{it}p$, $p \in M$, $\mathcal{V}$ be the CR structure of $M \subset L$,
Circle bundle?

For $p \in M$: $O_p = \text{orbit of } p$, say $p \sim p'$ iff $p' \in \overline{O}_p$.
let $\varphi : M \to B := M/\sim$.

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The closure of $\alpha_t$ in $\text{Iso}(M)$ is isomorphic to a torus $\mathbb{T}$.

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\[ M \text{ is a closed manifold, } \mathcal{L}_T g = 0. \text{ We only care that such } g \text{ exists} \]

$T$ is a nowhere vanishing real vector field preserving some Riemannian metric $g$
\[ \alpha_t \text{ is the flow of } T, \text{ is an invariant smooth density} \]
$E \to M$ is a Hermitian vector bundle, metric $h$
\[ \mathcal{A}_t : E \to E \text{ is a unitary bundle homomorphism covering } \alpha_t \]
Lie derivative:
\[ \phi \text{ section of } E, p \in M. t \mapsto \mathcal{A}_{-t}(\phi(\alpha_t(p))) \text{ is a curve in } E_p, \]
\[ \mathcal{L}_T(\phi)(p) = \frac{d}{dt} \bigg|_{t=0} \mathcal{A}_{-t}(\phi(\alpha_t(p))). \]

$P$ is a differential operator $C^\infty(M; E) \to C^\infty(M; E)$. We ask of order $m$
\[ [\mathcal{L}_T, P] = 0, [P, P^*] = 0 \]
\[ \sigma(P) + \sigma(-i\mathcal{L}_T)^m - \lambda I \text{ is invertible if } \lambda \in \Lambda. \]

Let $\mathcal{D} = \{ \phi \in H : \mathcal{L}_T \phi \in H \}$.
\[ -i\mathcal{L}_T|_\varphi : \mathcal{D} \subset H \to H \]
is selfadjoint with compact resolvent, in particular Fredholm.

\[ a \text{ hypersurface in the complex manifold } L \]
\[ M = \{ \eta \in L : |\eta|^2 = 1 \} \text{ for some Hermitian metric} \]

Example. Let $B$ be a compact complex manifold, $\varphi : M \to B$ the circle bundle
of a holomorphic line bundle $L \to B$, $T$ the generator of $t \mapsto e^{it}p$, $p \in M$,
$\mathcal{V}$ be the CR structure of $M \subset L$, $P$ be the Kohn Laplacian acting on $C^\infty(\Lambda^q \overline{\mathcal{V}}^*)$
$(0, q)$-forms on $M$.
Circle bundle?

For \( p \in M \): \( \mathcal{O}_p = \) orbit of \( p \), say \( p \sim p' \) iff \( p' \in \overline{\mathcal{O}_p} \).

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\[ P + (-i \mathcal{L}_T)^2 \text{ is elliptic, } \ker P = \bigoplus_{m \in \mathbb{Z}} \mathcal{E}_m, -i \mathcal{L}_T \varphi = m \varphi. \]
Circle bundle?

For \( p \in M \): \( O_p = \text{orbit of } p \), say \( p \sim p' \) iff \( p' \in \overline{O}_p \),
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The closure of \( \alpha_t \) in \( \text{Iso}(M) \) is isomorphic to a torus \( \mathbb{T} \).

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Example. Let \( B \) be a compact complex manifold, \( \varphi : M \rightarrow B \) the circle bundle
of a holomorphic line bundle \( L \rightarrow B \), \( T \) the generator of \( t \mapsto e^{it} p \), \( p \in M \),
\( \mathcal{N} \) be the CR structure of \( M \subset L \), \( P \) be the Kohn Laplacian acting on \( \mathcal{C}^\infty(\wedge^q \mathcal{N}^*) \)
\( P + (-iL_T)^2 \) is elliptic, \( \ker P = \bigoplus_{m \in \mathbb{Z}} \mathcal{E}_m \), \( -iL_T \phi = m \phi \).

(And there is an isomorphism \( \mathcal{E}_{-m} \approx H^{0,q}(M; L^\otimes m) \))
Hypoellipticity

Observe \( \sigma(-iL_T) = \tau I \) for some \( \tau : T^*M \to \mathbb{R} \).

\( M \) is a closed manifold, \( \mathcal{L}_T g = 0 \). We only care that such \( g \) exists.

\( T \) is a nowhere vanishing real vector field preserving some Riemannian metric \( g \).

\( \alpha_t \) is the flow of \( T \), \( \mathfrak{m} \) is an invariant smooth density.

\( E \to M \) is a Hermitian vector bundle, metric \( h \).

\( \mathfrak{A}_t : E \to E \) is a unitary bundle homomorphism covering \( \alpha_t \).

Lie derivative:

\[ \phi \text{ section of } E, \, p \in M. \, t \mapsto \mathfrak{A}_{-t}(\phi(a_t(p))) \text{ is a curve in } E_p, \]

\[ \mathcal{L}_T(\phi)(p) = \frac{d}{dt} \bigg|_{t=0} \mathfrak{A}_{-t}(\Phi(a_t(p))). \]

\( P \) is a differential operator \( C^\infty(M; E) \to C^\infty(M; E) \). We ask of order \( m \)

\[ [\mathcal{L}_T, P] = 0, \, [P, P^*] = 0 \] (so \( P + (-iL_T)^m \) is elliptic).

\[ \sigma(P) + \sigma(-iL_T)^m - \lambda I \text{ is invertible if } \lambda \in \Lambda. \]

Let \( \mathcal{D} = \{ \phi \in H : \mathcal{L}_T \phi \in H \}. \)

\[ -i\mathcal{L}_T |_{\mathcal{D}} : \mathcal{D} \subset H \to H \]

is selfadjoint with compact resolvent, in particular Fredholm.
Hypoellipticity

Observe $\sigma(-iL_T) = \tau I$ for some $\tau : T^*M \to \mathbb{R}$.

$M$ is a closed manifold, $\mathcal{L}_T g = 0$. We only care that such $g$ exists.

$T$ is a nowhere vanishing real vector field preserving some Riemannian metric $g$

$\alpha_t$ is the flow of $T$, $m$ is an invariant smooth density

$E \to M$ is a Hermitian vector bundle, metric $h$

$\mathfrak{U}_t : E \to E$ is a unitary bundle homomorphism covering $\alpha_t$

Lie derivative:

$\phi$ section of $E$, $p \in M$. $t \mapsto \mathfrak{U}_t(\phi(\alpha_t(p)))$ is a curve in $E_p$,

$$\mathcal{L}_T(\phi)(p) = \frac{d}{dt} \bigg|_{t=0} \mathfrak{U}_t(\phi(\alpha_t(p))).$$

$P$ is a differential operator $C^\infty(M; E) \to C^\infty(M; E)$. We ask of order $m$

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so $P + (-i\mathcal{L}_T)^m$ is elliptic

$\sigma(P) + \sigma(-i\mathcal{L}_T)^m - \lambda I$ is invertible if $\lambda \in \Lambda$.

Let $\mathcal{D} = \{\phi \in H : \mathcal{L}_T \phi \in H\}$.

$$-i\mathcal{L}_T|_\mathcal{D} : \mathcal{D} \subset H \to H$$

is selfadjoint with compact resolvent, in particular Fredholm.

$$\lim_{s \to \infty} \frac{1}{s} e^{-isf} \frac{1}{i} \mathcal{L}_T(e^{isf} \phi) = \langle df, T \rangle \phi$$

so $\tau(\xi) = \langle \xi, T \rangle$.
Hypoellipticity

Observe $\sigma(-i\mathcal{L}_T) = \tau I$ for some $\tau : T^* M \to \mathbb{R}$.

Invertibility of $\sigma(P) + \sigma((-i\mathcal{L}_T)^m)$ gives $\tau \neq 0$ on $\text{Char}(P)$:

$M$ is a closed manifold, $\mathcal{L}_T g = 0$. We only care that such $g$ exists

$T$ is a nowhere vanishing real vector field preserving some Riemannian metric $g$

$a_t$ is the flow of $T$, $m$ is an invariant smooth density

$E \to M$ is a Hermitian vector bundle, metric $h$

$\mathcal{A}_t : E \to E$ is a unitary bundle homomorphism covering $a_t$

Lie derivative:

$\phi$ section of $E$, $p \in M$. $t \mapsto \mathcal{A}_{-t}(\phi(a_t(p)))$ is a curve in $E_p$,

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$P$ is a differential operator $C^\infty(M; E) \to C^\infty(M; E)$. We ask

$[\mathcal{L}_T, P] = 0$, $[P, P^*] = 0$ (so $P + (-i\mathcal{L}_T)^m$ is elliptic)

$\sigma(P) + \sigma(-i\mathcal{L}_T)^m - \lambda I$ is invertible if $\lambda \in \Lambda$.

Let $\mathcal{D} = \{ \phi \in H : \mathcal{L}_T \phi \in H \}$.

$\left. -i\mathcal{L}_T \right|_{\mathcal{D}} : \mathcal{D} \subset H \to H$

is selfadjoint with compact resolvent, in particular Fredholm.

$$\lim_{s \to \infty} \frac{1}{s} e^{-isf} \frac{1}{i} \mathcal{L}_T (e^{isf} \phi) = \langle df, T \rangle \phi$$

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Hypoellipticity

Observe \( \sigma(-i\mathcal{L}_T) = \tau I \) for some \( \tau : T^*M \to \mathbb{R} \).

Invertibility of

\[ \sigma(P) + \sigma((-i\mathcal{L}_T)^m) \]

gives \( \tau \neq 0 \) on \( \text{Char}(P) \):

\[ \text{Char}(P) = \text{Char}^+(P) \cup \text{Char}^-(P) \]

according to \( \tau > 0 \) or \( \tau < 0 \)

\( M \) is a closed manifold, \( \mathcal{L}_T g = 0 \). We only care that such \( g \) exists

\( T \) is a nowhere vanishing real vector field preserving some Riemannian metric \( g \)

\( \alpha_t \) is the flow of \( T \), \( m \) is an invariant smooth density

\( E \to M \) is a Hermitian vector bundle, metric \( h \)

\( \mathcal{A}_t : E \to E \) is a unitary bundle homomorphism covering \( \alpha_t \)

Lie derivative:

\( \phi \) section of \( E, p \in M. t \mapsto \mathcal{A}_t(\phi(\alpha_t(p))) \) is a curve in \( E_p, \)

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\( P \) is a differential operator \( C^\infty(M; E) \to C^\infty(M; E) \). We ask

\[ [\mathcal{L}_T, P] = 0, [P, P^*] = 0 \]

so \( P + (-i\mathcal{L}_T)^m \) is elliptic

\[ \sigma(P) + \sigma((-i\mathcal{L}_T)^m) - \lambda I \text{ is invertible if } \lambda \in \Lambda. \]

Let \( \mathfrak{D} = \{ \phi \in H : \mathcal{L}_T \phi \in H \} \).

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**Hypoellipticity**

Observe $\sigma(-i \mathcal{L}_T) = \tau I$ for some $\tau : T^* M \to \mathbb{R}$.

Invertibility of

$\sigma(P) + \sigma((-i \mathcal{L}_T)^m)$

gives $\tau \neq 0$ on $\text{Char}(P)$:

$\text{Char}(P) = $ $\text{Char}^+(P) \cup \text{Char}^-(P)$

according to $\tau > 0$ or $\tau < 0$

**Theorem.** If $P$ is hypoelliptic on $\text{Char}^+(P)$ then $-i \mathcal{L}_T \big|_{\mathcal{D}}$ is semibounded from above.

$M$ is a closed manifold, 
$\mathcal{L}_T g = 0$. We only care that such $g$ exists

$T$ is a nowhere vanishing real vector field preserving some Riemannian metric $g$

$\alpha_t$ is the flow of $T$, $m$ is an invariant smooth density

$E \to M$ is a Hermitian vector bundle, metric $h$

$\mathfrak{A}_t : E \to E$ is a unitary bundle homomorphism covering $\alpha_t$

Lie derivative:

$\phi$ section of $E$, $p \in M$. $t \mapsto \mathfrak{A}_t(\phi(\alpha_t(p)))$ is a curve in $E_p$,

$\mathcal{L}_T(\phi)(p) = \frac{d}{dt} \big|_{t=0} \mathfrak{A}_t(\phi(\alpha_t(p)))$.

$P$ is a differential operator $C^\infty(M; E) \to C^\infty(M; E)$. We ask

$[\mathcal{L}_T, P] = 0$, $[P, P^*] = 0$ (so $P + (-i \mathcal{L}_T)^m$ is elliptic)

$\sigma(P) + \sigma((-i \mathcal{L}_T)^m - \lambda I$ is invertible if $\lambda \in \Lambda$.

Let $\mathcal{D} = \{ \phi \in H : \mathcal{L}_T \phi \in H \}$.

$-i \mathcal{L}_T \big|_{\mathcal{D}} : \mathcal{D} \subset H \to H$

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Hypoellipticity

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$P$ is a differential operator $C^\infty(M; E) \to C^\infty(M; E)$. We ask

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Let $\mathcal{D} = \{ \phi \in H : \mathcal{L}_T \phi \in H \}$.

$$-i\mathcal{L}_T|_\mathcal{D} : \mathcal{D} \subset H \to H$$

is selfadjoint with compact resolvent, in particular Fredholm.

(At most finitely many positive elements in spectrum.)
Hypoellipticity

Observe $\sigma(-i\mathcal{L}_T) = \tau I$ for some $\tau : T^* M \to \mathbb{R}$.

Invertibility of

$\sigma(P) + \sigma((-i\mathcal{L}_T)^m)$

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$\text{Char}(P) = \text{Char}^+(P) \cup \text{Char}^-(P)$

according to $\tau > 0$ or $\tau < 0$

Theorem. If $P$ is hypoelliptic on $\text{Char}^+(P)$ then $-i\mathcal{L}_T$ above. If $M$ carries an invariant Levi non-degenerate CR structure then the Kohn Laplacians are hypoelliptic in degree $q \neq q^\pm$. 

$M$ is a closed manifold, $\mathcal{L}_T g = 0$. We only care that such $g$ exists

$T$ is a nowhere vanishing real vector field preserving some Riemannian metric $g$

$\alpha_t$ is the flow of $T$, $m$ is an invariant smooth density $E \to M$ is a Hermitian vector bundle, metric $h$

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Lie derivative:

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$P$ is a differential operator $C^\infty(M; E) \to C^\infty(M; E)$. We ask

$[\mathcal{L}_T, P] = 0$, $[P, P^*] = 0$

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$\sigma(P) + \sigma(-i\mathcal{L}_T)^m - \lambda I$ is invertible if $\lambda \in \Lambda$.

Let $\mathcal{D} = \{ \phi \in H : \mathcal{L}_T \phi \in H \}$.

$-i\mathcal{L}_T \big|_{\mathcal{D}} : \mathcal{D} \subset H \to H$

is selfadjoint with compact resolvent, in particular Fredholm.

is semibounded from

(At most finitely many positive elements in spectrum.)

$q^\pm = \#\text{pos/neg Levi eigenvalues}.}$
Hypoellipticity

Observe $\sigma(-i\mathcal{L}_T) = \tau I$ for some $\tau : T^*M \to \mathbb{R}$.

Invertibility of

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$$\text{Char}(P) = \text{Char}^+(P) \cup \text{Char}^-(P)$$

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**Theorem.** If $P$ is hypoelliptic on $\text{Char}^+(P)$ then $-i\mathcal{L}_T$ above. If $M$ carries an invariant Levi non-degenerate CR structure then the Kohn Laplacians are hypoelliptic in degree $q \neq q^\pm$.

$M$ is a closed manifold, $\mathcal{L}_T g = 0$. We only care that such $g$ exists $T$ is a nowhere vanishing real vector field preserving some Riemannian metric $g$

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Lie derivative:

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$\mathcal{D}$ is semibounded from (At most finitely many positive elements in spectrum.)

$q^\pm = \# \text{pos/neg Levi eigenvalues.}$

by work of Boutet de Monvel & Sjöstrand, '70s
**Hypoellipticity**

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**Theorem.** If $P$ is hypoelliptic on $\text{Char}^+(P)$ then $-i\mathcal{L}_T$ above. If $M$ carries an invariant Levi non-degenerate CR structure then the Kohn Laplacians are hypoelliptic in degree $q \neq q^\pm$. So finite spectrum in these degrees

$M$ is a closed manifold, $\mathcal{L}_T g = 0$. We only care that such $g$ exists

$T$ is a nowhere vanishing real vector field preserving some Riemannian metric $g$

$\alpha_t$ is the flow of $T$, $m$ is an invariant smooth density

$E \to M$ is a Hermitian vector bundle, metric $h$

$\mathcal{A}_t : E \to E$ is a unitary bundle homomorphism covering $\alpha_t$

Lie derivative:

$\phi$ section of $E$, $p \in M$. $t \mapsto \mathcal{A}_{-t}(\phi(\alpha_t(p)))$ is a curve in $E_p$,

$\mathcal{L}_T(\phi)(p) = \left. \frac{d}{dt} \right|_{t=0} \mathcal{A}_{-t}(\Phi(\alpha_t(p)))$.

$P$ is a differential operator $C^\infty(M; E) \to C^\infty(M; E)$. We ask

$[\mathcal{L}_T, P] = 0$, $[P, P^*] = 0$ (so $P + (-i\mathcal{L}_T)^m$ is elliptic)

$\sigma(P) + \sigma((-i\mathcal{L}_T)^m - \lambda I$ is invertible if $\lambda \in \Lambda$.

Let $\mathcal{D} = \{ \phi \in H : \mathcal{L}_T \phi \in H \}$.

$-i\mathcal{L}_T|_{\mathcal{D}} : \mathcal{D} \subset H \to H$

is selfadjoint with compact resolvent, in particular Fredholm.
**Hypoellipticity**

Observe $\sigma(-i\mathcal{L}_T) = \tau I$ for some $\tau : T^*M \to \mathbb{R}$.

Invertibility of
$$\sigma(P) + \sigma((-i\mathcal{L}_T)^m)$$
gives $\tau \neq 0$ on $\text{Char}(P)$:
$$\text{Char}(P) = \text{Char}^+(P) \cup \text{Char}^-(P)$$
according to $\tau > 0$ or $\tau < 0$

**Theorem.** If $P$ is hypoelliptic on $\text{Char}^+(P)$ then $-i\mathcal{L}_T$ above. If $M$ carries an invariant Levi non-degenerate CR structure then the Kohn Laplacians are hypoelliptic in degree $q \neq q^\pm$. So finite spectrum in these degrees

**Example.** Let $B$ be a compact complex manifold, $\varphi : M \to B$ the circle bundle of a holomorphic line bundle $L \to B$, $T$ the generator of $t \mapsto e^{it}p, p \in M$, $\mathcal{V}$ be the CR structure of $M \subset L$, $P$ be the Kohn Laplacian acting on $C^\infty(\Lambda^q\mathcal{V}^*)$ ($0,q$)-forms on $M$

$$P + (-i\mathcal{L}_T)^2$$
is elliptic, $\ker P = \bigoplus_{m \in \mathbb{Z}} \mathcal{E}_m, -i\mathcal{L}_T\phi = m\phi$.

(And there is an isomorphism $\mathcal{E}_{-m} \approx H^{0,q}(M;L^{-m}))$
Proofs

$M$ is a closed manifold, $\mathcal{L}_T g = 0$. We only care that such $g$ exists

$T$ is a nowhere vanishing real vector field preserving some Riemannian metric $g$

$\alpha_t$ is the flow of $T$, $m$ is an invariant smooth density

$E \to M$ is a Hermitian vector bundle, metric $h$

$\mathfrak{A}_t : E \to E$ is a unitary bundle homomorphism covering $\alpha_t$

Lie derivative:

For a section of $E$, $p \in M$, $t \mapsto \mathfrak{A}_t^{-1}(\phi(\alpha_t(p)))$ is a curve in $E_p$,

$\mathcal{L}_T(\phi)(p) = \frac{d}{dt} \bigg|_{t=0} \mathfrak{A}_t^{-1}(\Phi(\alpha_t(p))).$


$P$ is a differential operator $C^\infty(M; E) \to C^\infty(M; E)$. We ask

$[\mathcal{L}_T, P] = 0$, $[P, P^*] = 0$

(then $P + (-i\mathcal{L}_T)^m$ is elliptic)

$\sigma(P) + \sigma(-i\mathcal{L}_T)^m - \lambda I$ is invertible if $\lambda \in \Lambda$

Let $\mathcal{D} = \{ \phi \in H : \mathcal{L}_T \phi \in H \}$.

$-i\mathcal{L}_T |_\mathcal{D} : \mathcal{D} \subset H \to H$

is selfadjoint with compact resolvent, in particular Fredholm.

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Parametrix:

Suppose $Q$ is a $T$-invariant parametrix for $P + (-i\mathcal{L}_T)^m$. 

Proofs

$M$ is a closed manifold, \( \mathcal{L} g = 0 \). We only care that such \( g \) exists

\( T \) is a nowhere vanishing real vector field preserving some Riemannian metric \( g \)
\( \alpha_t \) is the flow of \( T \), \( m \) is an invariant smooth density
\( E \rightarrow M \) is a Hermitian vector bundle, metric \( h \)
\( \mathfrak{A}_t : E \rightarrow E \) is a unitary bundle homomorphism covering \( \alpha_t \)
Lie derivative:

\[ \phi \text{ section of } E, \ p \in M. \ t \mapsto \mathfrak{A}_{-t}(\phi(\alpha_t(p))) \text{ is a curve in } E_p, \]
\[ \mathcal{L}_T(\phi)(p) = \frac{d}{dt} \bigg|_{t=0} \mathfrak{A}_{-t}(\phi(\alpha_t(p))). \]

\( P \) is a differential operator \( C^\infty(M; E) \rightarrow C^\infty(M; E) \). We ask
of order \( m \)
\[ [\mathcal{L}_T, P] = 0, \ [P, P^*] = 0 \]
(\( \sigma(P) + \sigma(-i\mathcal{L}_T)^m - \lambda I \) is invertible if \( \lambda \in \Lambda \).

Let \( \mathcal{D} = \{ \phi \in H : \mathcal{L}_T \phi \in H \} \).
\[ -i\mathcal{L}_T \big|_\mathcal{D} : \mathcal{D} \subset H \rightarrow H \]
is selfadjoint with compact resolvent, in particular Fredholm.

### Parametrix:

Suppose \( Q \) is a \( T \)-invariant parametrix for \( P + (-i\mathcal{L}_T)^m \). Then
\[ \phi - R \phi = Q(P + (-i\mathcal{L}_T)^m) \phi \]
Proofs

$M$ is a closed manifold, $\mathcal{L}_T g = 0$. We only care that such $g$ exists.

$T$ is a nowhere vanishing real vector field preserving some Riemannian metric $g$

$\alpha_t$ is the flow of $T$, $m$ is an invariant smooth density

$E \to M$ is a Hermitian vector bundle, metric $h$

$\pi_t : E \to E$ is a unitary bundle homomorphism covering $\alpha_t$

Lie derivative:

$\phi$ section of $E$, $p \in M$. $t \mapsto \pi_t(\phi(\alpha_t(p)))$ is a curve in $E_p$,

$$\mathcal{L}_T(\phi)(p) = \frac{d}{dt} \bigg|_{t=0} \pi_t(\phi(\alpha_t(p))).$$

$P$ is a differential operator $C^\infty(M; E) \to C^\infty(M; E)$. We ask of order $m$

$[\mathcal{L}_T, P] = 0$, $[P, P^*] = 0$ (so $P + (-i\mathcal{L}_T)^m$ is elliptic)

$\sigma(P) + \sigma(-i\mathcal{L}_T)^m - \lambda I$ is invertible if $\lambda \in \Lambda$.

Let $\mathcal{D} = \{ \phi \in H : \mathcal{L}_T \phi \in H \}$.

$$-i\mathcal{L}_T|_{\mathcal{D}} : \mathcal{D} \subset H \to H$$

is selfadjoint with compact resolvent, in particular Fredholm.

Parametrix:

Suppose $Q$ is a $T$-invariant parametrix for $P + (-i\mathcal{L}_T)^m$. Then

$$\phi - R\phi = Q(P + (-i\mathcal{L}_T)^m)\phi = Q(-i\mathcal{L}_T)^m \phi$$

if $P\phi = 0$.
Proofs

$M$ is a closed manifold, $\mathcal{L}_T g = 0$. We only care that such $g$ exists

$T$ is a nowhere vanishing real vector field preserving some Riemannian metric $g$

$\alpha_t$ is the flow of $T$, $m$ is an invariant smooth density

$E \to M$ is a Hermitian vector bundle, metric $h$

$\mathcal{A}_t : E \to E$ is a unitary bundle homomorphism covering $\alpha_t$

Lie derivative:

$\phi$ section of $E$, $p \in M$. $t \mapsto \mathcal{A}_t^{-1}(\phi(\alpha_t(p)))$ is a curve in $E_p$,

$$\mathcal{L}_T(\phi)(p) = \frac{d}{dt} \big|_{t=0} \mathcal{A}_t^{-1}(\Phi(\alpha_t(p))).$$

$P$ is a differential operator $C^\infty(M; E) \to C^\infty(M; E)$. We ask

$[\mathcal{L}_T, P] = 0$, $[P, P^*] = 0$

$\sigma(P) + \sigma(-i\mathcal{L}_T)^m - \lambda I$ is invertible if $\lambda \in \Lambda$.

Let $\mathcal{D} = \{ \phi \in H : \mathcal{L}_T \phi \in H \}$.

$$-i\mathcal{L}_T|_\mathcal{D} : \mathcal{D} \subset H \to H$$

is selfadjoint with compact resolvent, in particular Fredholm.

Parametrix:

Suppose $Q$ is a $T$-invariant parametrix for $P + (-i\mathcal{L}_T)^m$. Then

$$\phi - R\phi = Q(P + (-i\mathcal{L}_T)^m)\phi = Q(-i\mathcal{L}_T)^m\phi = [Q(-i\mathcal{L}_T)^{m-1}][-i\mathcal{L}_T]\phi$$

if $P\phi = 0$
Proofs

\( M \) is a closed manifold, \( \mathcal{L}_T g = 0 \). We only care that such \( g \) exists.

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\( E \to M \) is a Hermitian vector bundle, metric \( h \)

\( \mathfrak{A}_t : E \to E \) is a unitary bundle homomorphism covering \( \alpha_t \)

Lie derivative:

\( \phi \) section of \( E \), \( p \in M \). \( t \mapsto \mathfrak{A}_{-t} (\phi(\alpha_t(p))) \) is a curve in \( E_p \),

\[ \mathcal{L}_T(\phi)(p) = \left. \frac{d}{dt} \right|_{t=0} \mathfrak{A}_{-t}(\Phi(\alpha_t(p))). \]

\( P \) is a differential operator \( C^\infty(M; E) \to C^\infty(M; E) \). We ask

of order \( m \)

\[ [\mathcal{L}_T, P] = 0, [P, P^*] = 0 \] (so \( P + (-i\mathcal{L}_T)^m \) is elliptic)

\[ \sigma(P) + \sigma(-i\mathcal{L}_T)^m - \lambda I \text{ is invertible if } \lambda \in \Lambda. \]

Let \( \mathcal{D} = \{ \phi \in H : \mathcal{L}_T \phi \in H \} \).

\[ -i\mathcal{L}_T \big|_\mathcal{D} : \mathcal{D} \subset H \to H \]

is selfadjoint with compact resolvent, in particular Fredholm.

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Parametrix:

Suppose \( Q \) is a \( T \)-invariant parametrix for \( P + (-i\mathcal{L}_T)^m \). Then

\[ \phi - R\phi = Q(P + (-i\mathcal{L}_T)^m)\phi = Q(-i\mathcal{L}_T)^m\phi = [Q(-i\mathcal{L}_T)^{m-1}]( -i\mathcal{L}_T)\phi \]

if \( P\phi = 0 \)

parametrix, compact
**Proofs**

$M$ is a closed manifold, $\mathcal{L}_T g = 0$. We only care that such $g$ exists.

$T$ is a nowhere vanishing real vector field preserving some Riemannian metric $g$

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$\mathcal{A}_t : E \to E$ is a unitary bundle homomorphism covering $\alpha_t$

Lie derivative:

\[ \phi \text{ section of } E, \ p \in M. \ t \mapsto \mathcal{A}_{-t}(\phi(\alpha_t(p))) \text{ is a curve in } E_p, \]

\[ \mathcal{L}_T(\phi)(p) = \frac{d}{dt} \bigg|_{t=0} \mathcal{A}_{-t}(\Phi(\alpha_t(p))). \]

$P$ is a differential operator $C^\infty(M; E) \to C^\infty(M; E)$. We ask

$[\mathcal{L}_T, P] = 0$, $[P, P^*] = 0$

$\sigma(P) + \sigma(-i\mathcal{L}_T)^m - \lambda I$ is invertible if $\lambda \in \Lambda$.

Let $\mathcal{D} = \{\phi \in H : \mathcal{L}_T \phi \in H\}$.

$-i\mathcal{L}_T |_{\mathcal{D}} : \mathcal{D} \subset H \to H$

is selfadjoint with compact resolvent, in particular Fredholm.

**Parametrix:**

Suppose $Q$ is a $T$-invariant parametrix for $P + (-i\mathcal{L}_T)^m$. Then

\[ \phi - R\phi = Q(P + (-i\mathcal{L}_T)^m)\phi = Q(-i\mathcal{L}_T)^m\phi = [Q(-i\mathcal{L}_T)^{m-1}](-i\mathcal{L}_T)\phi \]

if $P\phi = 0$

**Symmetry:**

\[ d(h(\phi, \psi) T|_m) \int_M d(h(\phi, \psi) T|_m) = 0 \]
Proofs

\( M \) is a closed manifold, \( \mathcal{L}_T g = 0 \). We only care that such \( g \) exists.

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Lie derivative:

- \( \phi \) section of \( E \), \( p \in M \). \( t \mapsto \mathcal{A}_{-t}(\phi(\alpha_t(p))) \) is a curve in \( E_p \),

\[ \mathcal{L}_T(\phi)(p) = \frac{d}{dt} \bigg|_{t=0} \mathcal{A}_{-t}(\Phi(\alpha_t(p))). \]

\( P \) is a differential operator \( C^\infty(M; E) \to C^\infty(M; E) \). We ask

\[ [\mathcal{L}_T, P] = 0, \quad [P, P^*] = 0 \] (so \( P + (-i \mathcal{L}_T)^m \) is elliptic)

\[ \sigma(P) + \sigma(-i \mathcal{L}_T)^m - \lambda \mathbb{I} \text{ is invertible if } \lambda \in \Lambda. \]

Let \( \mathcal{D} = \{ \phi \in H : \mathcal{L}_T \phi \in H \} \).

\[ -i \mathcal{L}_T \big|_{\mathcal{D}} : \mathcal{D} \subset H \to H \]

is selfadjoint with compact resolvent, in particular Fredholm.

Parametrix:

Suppose \( Q \) is a \( T \)-invariant parametrix for \( P + (-i \mathcal{L}_T)^m \). Then

\[ \phi - R\phi = Q(P + (-i \mathcal{L}_T)^m)\phi = Q(-i \mathcal{L}_T)^m\phi = [Q(-i \mathcal{L}_T)^{m-1}](-i \mathcal{L}_T)\phi \]

if \( P\phi = 0 \)

Symmetry:

\[ \mathcal{L}_T(h(\phi, \psi) m) = d(h(\phi, \psi)T|m) \int_M d(h(\phi, \psi)T|m) = 0 \]
Problems

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$M$ is a closed manifold, \( L_T g = 0 \). We only care that such \( g \) exists.

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Lie derivative:

\[
\phi \text{ section of } E, \ p \in M, \ t \mapsto \mathfrak{U}_{-t}(\phi(\alpha_t(p))) \text{ is a curve in } E_p,
\]

\[
\mathcal{L}_T(\phi)(p) = \left. \frac{d}{dt} \right|_{t=0} \mathfrak{U}_{-t}(\phi(\alpha_t(p))).
\]

\( P \) is a differential operator \( C^\infty(M; E) \to C^\infty(M; E) \). We ask of order \( m \)

\[
[L_T, P] = 0, \ [P, P^*] = 0 \quad \text{(so } P + (-iL_T)^m \text{ is elliptic)}
\]

\[\sigma(P) + \sigma(-iL_T)^m - \lambda I \text{ is invertible if } \lambda \in \Lambda.\]

Let \( \mathcal{D} = \{ \phi \in H : L_T \phi \in H \} \)

\[
-iL_T|_{\mathcal{D}} : \mathcal{D} \subset H \to H
\]

is selfadjoint with compact resolvent, in particular Fredholm.

---

**Proofs**

**Parametrix:**

Suppose \( Q \) is a \( T \)-invariant parametrix for \( P + (-iL_T)^m \). Then

\[
\phi - R\phi = Q(P + (-iL_T)^m)\phi = Q(-iL_T)^m\phi = \left[ Q(-iL_T)^{m-1} \right](-iL_T)\phi
\]

if \( P\phi = 0 \)

**Symmetry:**

\[
(h(L_T\phi, \psi) + h(\phi, L_T\psi)) m = L_T(h(\phi, \psi)m) = d(h(\phi, \psi)T|m)m
\]

Selfadjointness uses \( \sigma(P) + \sigma((-iL_T)^m) - \lambda I \) invertible for large \( \lambda \) (\( \Lambda \) is a ray of minimal growth) and formal normality of \( P \).
Proofs, details in (1)
$M$ is a closed manifold, \( \mathcal{L}_T g = 0 \). We only care that such \( g \) exists.

\( T \) is a nowhere vanishing real vector field preserving some Riemannian metric \( g \)

\( a_t \) is the flow of \( T \), \( m \) is an invariant smooth density

\( E \to M \) is a Hermitian vector bundle, metric \( h \)

\( \mathfrak{A}_t : E \to E \) is a unitary bundle homomorphism covering \( a_t \)

Lie derivative:

\[
\phi \text{ section of } E, \ p \in M. \ t \mapsto \mathfrak{A}_{-t}(\phi(a_t(p))) \text{ is a curve in } E_p, \\
\mathcal{L}_T(\phi)(p) = \frac{d}{dt} \bigg|_{t=0} \mathfrak{A}_{-t}(\Phi(a_t(p))).
\]

\( P \) is a differential operator \( C^\infty(M; E) \to C^\infty(M; E) \). We ask of order \( m \)

\[
[\mathcal{L}_T, P] = 0, \ [P, P^*] = 0 \quad (\text{so } P + (-i\mathcal{L}_T)^m \text{ is elliptic})
\]

\( \sigma(P) + \sigma(-i\mathcal{L}_T)^m - \lambda I \text{ is invertible if } \lambda \in \Lambda. \)

Let \( \mathcal{D} = \{ \phi \in H : \mathcal{L}_T \phi \in H \} \).

\[
-i\mathcal{L}_T \big|_{\mathcal{D}} : \mathcal{D} \subset H \to H
\]

is selfadjoint with compact resolvent, in particular Fredholm.
Finiteness of Spectrum

$M$ is a closed manifold, $\mathcal{L}_T g = 0$. We only care that such $g$ exists.

$T$ is a nowhere vanishing real vector field preserving some Riemannian metric $g$

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$E \to M$ is a Hermitian vector bundle, metric $h$

$\mathcal{A}_t : E \to E$ is a unitary bundle homomorphism covering $\alpha_t$

Lie derivative:

- $\phi$ section of $E$, $p \in M$, $t \mapsto \mathcal{A}_{-t}(\phi(\alpha_t(p)))$ is a curve in $E_p$,
- $\mathcal{L}_T(\phi)(p) = \frac{d}{dt} \bigg|_{t=0} \mathcal{A}_{-t}(\phi(\alpha_t(p)))$.

$P$ is a differential operator $C^\infty(M; E) \to C^\infty(M; E)$. We ask

- $[\mathcal{L}_T, P] = 0$, $[P, P^*] = 0$ (so $P + (-i\mathcal{L}_T)^m$ is elliptic)
- $\sigma(P) + \sigma(-i\mathcal{L}_T)^m - \lambda I$ is invertible if $\lambda \in \Lambda$.

Let $\mathcal{D} = \{ \phi \in H : \mathcal{L}_T \phi \in H \}$.

- $-i\mathcal{L}_T|_\mathcal{D} : \mathcal{D} \subset H \to H$

is selfadjoint with compact resolvent, in particular Fredholm.

Suppose $P$ is hypoelliptic on $\tau > 0$ but there there is $\{ \tau_k \}_{k=1}^\infty$, $\{ \phi_k \}_{k=0}^\infty$

- $-i\mathcal{L}_T \phi_k = \tau_k \phi_k$, $\| \phi_k \| = 1$ (and $P \phi_k = 0$).
Finiteness of Spectrum

- $M$ is a closed manifold,
- $T$ is a nowhere vanishing real vector field preserving some Riemannian metric $g$
- $a_t$ is the flow of $T$, $m$ is an invariant smooth density
- $E \to M$ is a Hermitian vector bundle, metric $h$
- $\mathfrak{A}_t : E \to E$ is a unitary bundle homomorphism covering $a_t$
- Lie derivative:
  - $\phi$ section of $E$, $p \in M$. $t \mapsto \mathfrak{A}_{-t}(\phi(a_t(p)))$ is a curve in $E_p$,
  - $\mathcal{L}_T(\phi)(p) = \frac{d}{dt} \bigg|_{t=0} \mathfrak{A}_{-t}(\Phi(a_t(p)))$.

- $P$ is a differential operator $C^\infty(M; E) \to C^\infty(M; E)$. We ask of order $m$
  - $[\mathcal{L}_T, P] = 0$, $[P, P^*] = 0$ (so $P + (-i\mathcal{L}_T)^m$ is elliptic)
  - $\sigma(P) + \sigma(-i\mathcal{L}_T)^m - \lambda I$ is invertible if $\lambda \in \Lambda$.

Let $\mathcal{D} = \{ \phi \in H : \mathcal{L}_T \phi \in H \}$.

- $-i\mathcal{L}_T \bigg|_{\mathcal{D}} : \mathcal{D} \subset H \to H$
- is selfadjoint with compact resolvent, in particular Fredholm.

Suppose $P$ is hypoelliptic on $\tau > 0$ but there there is $\{ T_k \}_{k=1}^\infty, \{ \phi_k \}_{k=0}^\infty$

- $-i\mathcal{L}_T \phi_k = T_k \phi_k$, $\| \phi_k \| = 1$ (and $P \phi_k = 0$).

Then $\phi = \sum \phi_k$ is a distribution such that $P \phi = 0$.
Finiteness of Spectrum

\[ M \text{ is a closed manifold,} \quad \mathcal{L}_T g = 0. \text{ We only care that such } g \text{ exists} \]

\[ T \text{ is a nowhere vanishing real vector field preserving some Riemannian metric } g \]

\[ a_t \text{ is the flow of } T, \text{ m is an invariant smooth density} \]

\[ E \rightarrow M \text{ is a Hermitian vector bundle, metric } h \]

\[ \mathcal{A}_t : E \rightarrow E \text{ is a unitary bundle homomorphism covering } a_t \]

\[ \text{Lie derivative:} \]

\[ \phi \text{ section of } E, \quad p \in M. \quad t \mapsto \mathcal{A}_t(\phi(a_t(p))) \text{ is a curve in } E_p, \]

\[ \mathcal{L}_T(\phi)(p) = \frac{d}{dt} \bigg|_{t=0} \mathcal{A}_t(\phi(a_t(p))). \]

\[ P \text{ is a differential operator } C^\infty(M; E) \rightarrow C^\infty(M; E). \text{ We ask} \]

\[ [\mathcal{L}_T, P] = 0, \quad [P, P^*] = 0 \quad \text{(so } P + (-i\mathcal{L}_T)^m \text{ is elliptic)} \]

\[ \sigma(P) + \sigma(-i\mathcal{L}_T)^m - \lambda I \text{ is invertible if } \lambda \in \Lambda. \]

\[ \text{Let } \mathcal{D} = \{\phi \in H : \mathcal{L}_T \phi \in H\}. \]

\[ -i\mathcal{L}_T \big|_{\mathcal{D}} : \mathcal{D} \subset H \rightarrow H \]

\[ \text{is selfadjoint with compact resolvent, in particular Fredholm.} \]

Suppose \( P \) is hypoelliptic on \( \tau > 0 \) but there there is \( \{\tau_k\}_{k=1}^\infty, \{\phi_k\}_{k=0}^\infty \)

\[ -i\mathcal{L}_T \phi_k = \tau_k \phi_k, \quad ||\phi_k|| = 1 \quad \text{(and } P\phi_k = 0). \]

Then \( \phi = \sum \phi_k \) is a distribution such that \( P\phi = 0. \)

Let \( p_0 \in M, S \) a piece of a hypersurface through \( p \) transversal to \( T \).
Finiteness of Spectrum

\[ M \text{ is a closed manifold, } \quad L_T g = 0. \text{ We only care that such } g \text{ exists} \]

\( T \) is a nowhere vanishing real vector field preserving some Riemannian metric \( g \)

\( \alpha_t \) is the flow of \( T \), \( m \) is an invariant smooth density

\( E \to M \) is a Hermitian vector bundle, metric \( h \)

\( \mathcal{A}_t : E \to E \) is a unitary bundle homomorphism covering \( \alpha_t \)

Lie derivative:

\( \phi \) section of \( E, \ p \in M. \ t \mapsto \mathcal{A}_{-t}(\phi(\alpha_t(p))) \) is a curve in \( E_p \),

\[ L_T(\phi)(p) = \frac{d}{dt} \bigg|_{t=0} \mathcal{A}_{-t}(\Phi(\alpha_t(p))). \]

\( P \) is a differential operator \( C^\infty(M; E) \to C^\infty(M; E) \). We ask of order \( m \)

\[ [L_T, P] = 0, \quad [P, P^*] = 0 \] (so \( P + (-iL_T)^m \) is elliptic)

\( \sigma(P) + \sigma(-iL_T)^m - \lambda I \) is invertible if \( \lambda \in \Lambda \).

Let \( \mathcal{D} = \{ \phi \in H : L_T \phi \in H \} \).

\[ -iL_T|_{\mathcal{D}} : \mathcal{D} \subset H \to H \]

is selfadjoint with compact resolvent, in particular Fredholm.

Suppose \( P \) is hypoelliptic on \( \tau > 0 \) but there there is \( \{ \tau_k \}_{k=1}^\infty, \{ \phi_k \}_{k=0}^\infty \)

\[ -iL_T \phi_k = \tau_k \phi_k, \quad \| \phi_k \| = 1 \] (and \( P \phi_k = 0 \)).

Then \( \phi = \sum \phi_k \) is a distribution such that \( P \phi = 0 \).

Let \( p_0 \in M, S \) a piece of a hypersurface through \( p \) transversal to \( T \). If \( p \in S \), then

\[ \mathcal{A}_{-t}(\alpha_t(p)) = e^{i\tau_k t} \phi(p). \]
Suppose $P$ is hypoelliptic on $\tau > 0$ but there there is $\{\tau_k\}_{k=1}^{\infty}, \{\phi_k\}_{k=0}^{\infty}$

$$-i\mathcal{L}_T \phi_k = \tau_k \phi_k, \quad \|\phi_k\| = 1 \quad \text{(and } P\phi_k = 0).$$

Then $\phi = \sum \phi_k$ is a distribution such that $P\phi = 0$.

Let $p_0 \in M$, $S$ a piece of a hypersurface through $p$ transversal to $\mathcal{T}$. If $p \in S$, then

$$\mathcal{U}_{-t}\phi(a_t(p)) = e^{i\tau_k t} \phi(p).$$

If $\chi \in C^\infty$ has small support near $p_0$, then

$$(\chi(a_t(p))\phi)\tau = \sum_k \hat{\chi}(\tau - \tau_k)\phi_k(p)$$
$M$ is a closed manifold, \( \mathcal{L}_T g = 0 \). We only care that such \( g \) exists.

\( T \) is a nowhere vanishing real vector field preserving some Riemannian metric \( g \)
\( \alpha_t \) is the flow of \( T \), \( m \) is an invariant smooth density
\( E \to M \) is a Hermitian vector bundle, metric \( h \)
\( \mathcal{U}_t : E \to E \) is a unitary bundle homomorphism covering \( \alpha_t \)

Lie derivative:

\[ \phi \text{ section of } E, p \in M. t \mapsto \mathcal{U}_{-t}(\phi(\alpha_t(p))) \text{ is a curve in } E_p, \]

\[
\mathcal{L}_T(\phi)(p) = \frac{d}{dt} \bigg|_{t=0} \mathcal{U}_{-t}(\Phi(\alpha_t(p))).
\]

\( P \) is a differential operator \( C^\infty(M; E) \to C^\infty(M; E) \). We ask of order \( m \)

\[
[\mathcal{L}_T, P] = 0, [P, P^*] = 0 \quad \text{(so } P + (-i\lambda_T)^m \text{ is elliptic)}
\]

\[
\sigma(P) + \sigma(-i\lambda_T)^m - \lambda I \text{ is invertible if } \lambda \in \Lambda.
\]

Let \( \mathcal{D} = \{ \phi \in H : \mathcal{L}_T \phi \in H \} \).

\[
-i\mathcal{L}_T |_{\mathcal{D}} : \mathcal{D} \subset H \to H
\]

is selfadjoint with compact resolvent, in particular Fredholm.

**Suppose** \( P \) is hypoelliptic on \( \tau > 0 \) but there there is \( \{ \tau_k \}_{k=1}^\infty \), \( \{ \phi_k \}_{k=0}^\infty \)

\[
-i\mathcal{L}_T \phi_k = \tau_k \phi_k, \quad \| \phi_k \| = 1 \quad \text{(and } P \phi_k = 0).\]

Then \( \phi = \sum \phi_k \) is a distribution such that \( P \phi = 0 \).

Let \( p_0 \in M \), \( S \) a piece of a hypersurface through \( p \) transversal to \( T \). If \( p \in S \), then

\[ \mathcal{U}_{-t}(\alpha_t(p)) = e^{i\tau_k t} \phi(p). \]

If \( \chi \in C^\infty \) has small support near \( p_0 \), then

\[
(\chi(\alpha_t(p))\phi)\tau \quad = \quad \sum_k \hat{\chi}(\tau - \tau_k) \phi_k(p)
\]
M is a closed manifold, \( \mathcal{L}_T g = 0 \). We only care that such \( g \) exists.

\( T \) is a nowhere vanishing real vector field preserving some Riemannian metric \( g \), \( \alpha_t \) is the flow of \( T \), \( m \) is an invariant smooth density \( E \to M \) is a Hermitian vector bundle, metric \( h \), \( \mathcal{A}_t : E \to E \) is a unitary bundle homomorphism covering \( \alpha_t \). Lie derivative:

1. \( \phi \) section of \( E, p \in M. t \mapsto \mathcal{A}_t^{-1}(\phi(\alpha_t(p))) \) is a curve in \( E_p, \)
2. \( \mathcal{L}_T(\phi)(p) = \frac{d}{dt} \bigg|_{t=0} \mathcal{A}_t^{-1}(\phi(\alpha_t(p))). \)

\( P \) is a differential operator \( C^\infty(M; E) \to C^\infty(M; E). \) We ask

- \([\mathcal{L}_T, P] = 0, [P, P^*] = 0 \) (so \( P + (-i\mathcal{L}_T)^m \) is elliptic)
- \( \sigma(P) + \sigma((-i\mathcal{L}_T)^m - \lambda I) \) is invertible if \( \lambda \in \Lambda. \)

Let \( \mathcal{D} = \{ \phi \in H : \mathcal{L}_T \phi \in H \}. \)

\[-i\mathcal{L}_T \bigg|_\mathcal{D} : \mathcal{D} \subset H \to H \]
is selfadjoint with compact resolvent, in particular Fredholm.

Suppose \( P \) is hypoelliptic on \( \tau > 0 \) but there there is \( \{ \tau_k \}_{k=1}^\infty \), \( \{ \phi_k \}_{k=0}^\infty \)

\[-i\mathcal{L}_T \phi_k = \tau_k \phi_k, \quad ||\phi_k|| = 1 \quad \text{(and } P \phi_k = 0). \]

Then \( \phi = \sum \phi_k \) is a distribution such that \( P \phi = 0. \)

Let \( p_0 \in M, S \) a piece of a hypersurface through \( p \) transversal to \( T \). If \( p \in S \), then \( \mathcal{A}_t^{-1}(\phi(\alpha_t(p)) = e^{i\tau_k t} \phi(p). \)

If \( \chi \in C^\infty \) has small support near \( p_0 \), then

\[(\chi(\alpha_t(p)) \phi)(\tau) = \sum_k \hat{\chi}(\tau - \tau_k) \phi_k(p)\]

Hypoellipticity implies this is rapidly decreasing a \( \tau \to \infty. \)
Suppose $P$ is hypoelliptic on $\tau > 0$ but there there is $\{\tau_k\}_{k=1}^\infty, \{\phi_k\}_{k=0}^\infty$ 

$$-i\mathcal{L}_T \phi_k = \tau_k \phi_k, \quad \|\phi_k\| = 1 \quad \text{(and $P\phi_k = 0$)}.$$ 

Then $\phi = \sum \phi_k$ is a distribution such that $P\phi = 0$.

Let $p_0 \in M$, $S$ a piece of a hypersurface through $p$ transversal to $\mathcal{T}$. If $p \in S$, then $\mathcal{U}_{-t}(a_t(p)) = e^{i\tau_k t} \phi(p)$. If $\chi \in C^\infty$ has small support near $p_0$, then 

$$(\chi(a_t(p))\phi)(\tau) = \sum_k \hat{\chi}(\tau - \tau_k) \phi_k(p)$$ 

Hypoellipticity implies this is rapidly decreasing a $\tau \to \infty$. Conclude $\phi_k$ arbitrarily small near $p_0$ for large $k$, then arbitrarily small (large $k$) on $M$ by compactness.
Thank you