On the solvability of degenerate PDEs

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Outline.

Solvability of PDEs is still an open area, even after the solution of the Nirenberg-Treves conjecture. Here I give some results, joint work with Serena Federico, about solvability of operators with multiple characteristics. I will describe models whose characteristic set is quite degenerate (due to the cooperation of two kinds of degeneracies) for which one can still have L^2 (or "better") local solvability.

1. Solvability (local, H^s to $H^{s'}$).
2. Multiple characteristics.
3. The starting models.
4. Our models: (ST), (MT) and (MST).
5. Examples.
6. Remarks.

Solvability. P a m-th order pdo, smooth coefficients.

Def. Local solvability.

P is locally solvable at $x_0 \in \Omega$ if $\exists V \ni x_0$ such that for all $v \in C^\infty(\Omega)$ there is $u \in \mathscr{D}'(\Omega)$ for which $Pu = v$ in V.

Def. H^s to $H^{s'}$ local solvability near x_0:

Given $x_0 \in \Omega$, $s, s' \in \mathbb{R}$, $\exists$ cmpct $K \subset \Omega$, $x_0 \in \mathring{K} = U$, s.t. $\forall v \in H^s_{\mathrm{loc}}(\Omega)$ $\exists u \in H^{s'}_{\mathrm{loc}}(\Omega)$ satisfying $Pu = v$ in U. (When $\forall s$, $s' = s + m - r$, $r =$ loss of derivatives.)

Solvability estimate:

By using Hahn-Banach, existence of $C_K > 0$ such that

$$\|P^*\varphi\|_{-s'} C_K \geq \|\varphi\|_{-s}, \quad \forall \varphi \in C_0^\infty(K)$$

yields H^s to $H^{s'}$ local solvability.
Hence, one may look also at necessary conditions to obtain a priori estimate. When (say) $s' = 0$:

$$(PP^*\varphi, \varphi) \gtrsim \|\varphi\|^2_{-s}, \quad \forall \varphi \in C_0^\infty(K).$$

Multiple characteristics ($m \geq 2$). Work by:

- Mendoza and Uhlmann: $P = D_1 D_2 + Q_1(x, D)$, P is not locally solvable if $\mathrm{Im}\, q_1$ changes sign on bicharacteristics of symbols ξ_1 and ξ_2 (x_1, x_2 lines), at the double characteristics points $\{\xi_1 = \xi_2 = 0\}$. (Condition sub(P).)
- Corwin, Rothschild, Müller, Peloso, Ricci, De Mari (left invariant vector field on step-2 nilpotent group); Müller: necessary conditions (suff. some cases).
- Treves: $P = P_1 P_2 + iQ_1$, p_1, p_2 real princ. type, q_1 real, $p_1^2 + p_2^2 + \{p_1, p_2\}^2$ elliptic. (H^s to H^{s+1} loc. solv., loss 1.) Helffer: $P = P_1 P_2 + Q$; P_1, P_2, Q 1st-order (h.-e. with loss 1).
- Kannai, Colombini-Cordaro-Pernazza, Federico-P. (see below);
- J.J.Kohn: S-o-s of complex vector fields (div.-free, bracket condition).
- Parenti-P.: Semi-global solvability (propagation of singularities) for multiple transversal symplectic characteristics.
- Dencker: Operators of sub-principal type (involutive characteristic manifold), necessary conditions. (See also Popivanov.)
- Treves: real or complex vector fields with critical points.
- P^* hypoelliptic $\Longrightarrow$ P locally solvable (Beals-Fefferman, Akamatsu, Zuily, Boutet-Grigis-Helffer, Parenti-Rodino, Kohn, Parenti-P.).

Models: Kannai* and Colombini-Cordaro-Pernazza.

- Kannai: $P = D_{x_2} x_1 D_{x_2} + iD_{x_1}$. Then $P^* = D_{x_2} x_1 D_{x_2} - iD_{x_1}$ (Kannai's op.) is hypoelliptic and non-solvable near $x_1 = 0$. (Note: $D = -i\partial$ throughout.) Hence P is loc. solv.
- Colombini-Cordaro-Pernazza: (gen. of Kannai; also Beals-Fefferman, Zuily/Akamatsu, condt's for h.-e.)

$$P = X(x,D)^* f X(x,D) + iX_0(x,D) + a_0,$$

iX, iX_0 real smooth vector fields, $f \in C^\omega$ and $a_0 \in C^\infty$ real-valued, $X_0 \neq 0$ & transversal where f changes sign (suitable sense).

Thm. (Colombini-Cordaro-Pernazza)

$f(0) = 0$. Under the above assumptions, if $iX_0 \mathrm{sgn}(f) \geq 0$ (they call it the (ψ) condition) then L^2 to L^2 local solvability near 0.

Main point: obtain the a priori inequality

$$\|u\|_0 \lesssim \|P^* u\|_0, \quad \forall u \in C_0^\infty(\mathrm{neigh}(0)).$$

Question: Can one hope for a better regularity of the solution?

Our class(es) (Federico-P.).

On $\Omega \subset \mathbb{R}^n$ open we are given a system $(X_0, X_1, \ldots, X_{N+1})$

(MT) $$P_1 = \sum_{j=1}^{N} X_j^* f X_j + X_{N+1} + iX_0 + a_0.$$

(ST) $$P_2 = \sum_{j=1}^{N} X_j^* f_j X_j + X_{N+1} + a_0.$$

(MST) $$P_3 = \sum_{j=1}^{N} X_j^* f_j X_j + X_{N+1} + iX_0 + a_0.$$

- $X_0(x, D), X_1(x, D), \ldots, X_{N+1}(x, D)$ 1st-order homogeneous pdos, X_0, X_{N+1} real princ. symb., $X_1, \ldots, X_N$ real princ. symb. in (MT) and (MST), complex in (ST). Write $X_j(x, \xi) = \langle \alpha_j(x), \xi \rangle$, $\alpha_j \in C^\infty(\Omega; \mathbb{R}^n)$ or $\alpha_j \in C^\infty(\Omega; \mathbb{C}^n)$.
- $f, f_1, \ldots, f_N \in C^\infty(\Omega; \mathbb{R})$, such that $df\big|_{f^{-1}(0)} \neq 0$. $a_0 \in C^\infty(\Omega; \mathbb{C})$

Remark:

- This way we may study a degeneracy due to the interplay of the vanishing of f and the characteristic set $\Sigma = \bigcap_{j=0}^{N} X_j^{-1}(0)$, of the system of vector fields. Also: $\mathrm{sub}(P)(\varrho) = X_{N+1}(\varrho) + iX_0(\varrho)$, $\varrho \in X_1^{-1}(0) \cap \ldots \cap X_N^{-1}(0)$.
- Our class contains operators whose formal adjoint is not hypoelliptic (A, Z).

Mixed-Type: Hypotheses (H1), (H2), (H3), (H4) and (H5).

(H1) $iX_0 f\big|_{f^{-1}(0)} > 0$

(H2) $\forall K \subset \Omega\ \exists C > 0$ s.t. $\forall j,\ 1 \le j \le N+1$,

$$\{X_j, X_0\}(x,\xi)^2 \le C \sum_{k=0}^{N} X_k(x,\xi)^2, \quad \forall (x,\xi) \in K \times \mathbb{R}^n$$

(H3) $\forall K \subset \Omega\ \exists C > 0$ s.t., with $d_{X_0} = -i\mathrm{div}\alpha_0$,

$$|(\operatorname{Im} d_{X_0}(x)) X_{N+1}(x,\xi)| \le C \Big(\sum_{j=0}^{N} X_j(x,\xi)^2 \Big)^{1/2}, \quad \forall (x,\xi) \in K \times \mathbb{R}^n,$$

(H4) $\varrho \in \Sigma$, $V(\varrho) = \mathrm{Span}\{H_0(\varrho), \dots, H_N(\varrho)\}$, let $J = J(\varrho) \subset \{0, \dots, N\}$ s.t. $\{H_j(\varrho)\}_{j \in J}$ basis of $V(\varrho)$, $M(\varrho) = [\{X_j, X_{j'}\}(\varrho)]_{j,j' \in J}$.
(H4) is satisfied at $x_0 \in f^{-1}(0)$ if $\pi^{-1}(x_0) \cap \Sigma \neq \emptyset$ and

$$\operatorname{rank} M(\varrho) \ge 2, \quad \forall \varrho \in \pi^{-1}(x_0) \cap \Sigma.$$

Hence Melin: $\mathrm{Tr}^+ F_{\sum_{j=0}^{N} X_j^* X_j} > 0$ on $\pi^{-1}(x_0) \cap \Sigma$.

(H5) $\mathscr{L}_k(x) = \mathrm{Span}_{\mathbb{R}}\{iX_0, \dots, iX_N$ and commutators up to length k at $x\}$.
(HM5) is satisfied at $x_0 \in f^{-1}(0)$ if $\pi^{-1}(x_0) \cap \Sigma \neq \emptyset$ and $\exists r \ge 1$ s.t.

$$\dim \mathscr{L}_r(x_0) = n.$$

$$P_1 = \sum_{j=1}^{N} X_j^* f X_j + X_{N+1} + iX_0 + a_0.$$

Thm. (Solvability of P_1 near $S = f^{-1}(0)$; Federico-P.)

Supposing (H1), (H2) and (H3), one has:

(i) For all $x_0 \in S$ the operator P_1 is L^2 to L^2 loc. solv. near x_0;

(ii) If $x_0 \in S$ is s.t. $\pi^{-1}(x_0) \cap \Sigma \neq \emptyset$ and (H4) holds at x_0 then P_1 is $H^{-1/2}$ to L^2 loc. solv. near x_0;

(iii) If $x_0 \in S$ is s.t. $\pi^{-1}(x_0) \cap \Sigma \neq \emptyset$ and (H5) holds at x_0, some $r \geq 2$, then P_1 is $H^{-1/r}$ to L^2 loc. solv. near x_0;

(iv) If $x_0 \in S$ is s.t. $\pi^{-1}(x_0) \cap \Sigma = \emptyset$ then P_1 is H^{-1} to L^2 loc. solv. near x_0.

The Main Estimate:

$\exists K \subset \Omega$ suff. small with $x_0 \in \mathring{K}$, $\exists c_K, C_K > 0$ s.t.

$$2\,\mathrm{Re}\,(P_1^* u, -iX_0 u) \geq c_K \sum_{j=0}^{N} \|X_j u\|_0^2 + \frac{3}{2}\|X_0 u\|_0^2 - C_K \|u\|_0^2,\ \forall u \in C_0^\infty(K).$$

ME goes through Fefferman-Phong inequality:

Let $Y = -\mathrm{Re}\big((\mathrm{Im} d_{X_0})X_{N+1}\big)$, and $\widehat{P}_{\epsilon,\gamma} := \sum_{j=0}^{N}\Big(X_j^* X_j - \frac{\epsilon}{\gamma}[X_j, X_0]^*[X_j, X_0]\Big) + \frac{1}{\gamma_0} Y$, where $\epsilon = \|f\|_{L^\infty(K)} \to 0$ when $K \searrow \{x_0\}$. If (H2) and (H3) hold then may shrink K about x_0 so that (choice of ϵ, γ)

$$(\widehat{P}_{\epsilon,\gamma} u, u) \geq -C\|u\|_0^2,\ \forall u \in C_0^\infty(K).$$

Solvability estimate:

Gårding's/Melin's/Hörmander's hypoelliptic estimates (resp.) and Poincaré (on shrinking K) to reabsorb the L^2 error.

$$\|P^* u\|_0^2 \geq c_0 \|X_0 u\|_0^2 + c_1 \|u\|_s^2 - C_2 \|u\|_0^2,\ \forall u \in C_0^\infty(K)\ (s = 1,\ 1/2,\ 1/r,\ 0).$$

Examples.

Example 1.

In $\mathbb{R}^2$, $x = (x_1, x_2)$, $g(x_2) = 1 + x_2^2$, $f(x) = x_1 - (x_2 + x_2^3/3)$,

$$A(x) = \begin{bmatrix} g & 1 \\ 1 & 1/g \end{bmatrix} \geq 0, \quad \dim \operatorname{Ker} A(x) = 1, \ \forall x.$$

$$P = \sum_{j,j'=1}^{2} D_j \Big(f a_{jj'} D_{j'} \Big) + X_3 + iX_0 + a_0,$$

$$X(x,\xi) = g(x_2)\xi_1 + \xi_2, \ X_0(x,\xi) = \alpha\xi_1 + \frac{\xi_2}{g(x_2)}, \ X_3(x,\xi) = \mu_1(x)X(x,\xi) + \mu_2(x)X_0(x,\xi),$$

with $\alpha > 1$. Then $P = \sum_{j=1}^{2} X_j^* f X_j + X_3 + iX_0 + a_0$ with

$$X_1(x,\xi) = \sqrt{g(x_2)}\, \frac{X(x,\xi)}{\sqrt{1+g(x_2)^2}}, \quad X_2(x,\xi) = \frac{1}{\sqrt{g(x_2)}}\, \frac{X(x,\xi)}{\sqrt{1+g(x_2)^2}}.$$

Conditions (H1), (H2) and (H3) hold, but (H4) or (H5) does not ($\{X_0, X\} = \mu(x)X$). We have L^2 to L^2 local solvability near $f^{-1}(0)$.

Example 2.

In $\mathbb{R}^3_{x_1,x_2,x_3}$, let $k \geq 0$ be an integer, let $f(x) = x_2$,

$$X_1 = D_{x_1},\ X_2 = x_1^k D_{x_3},\ X_3 = \beta(x) D_{x_1},\ X_0 = D_{x_2},\ \beta \in C^\infty,$$

$$P = \sum_{j=1}^{2} X_j^* f X_j + X_3 + i X_0.$$

Then $d_{X_0} = 0$, (H1), (H2) ($\{X_0, X_j\} = 0$, $j = 1, 2$, $|\{X_0, X_3\}|^2 \lesssim X_1^2$) and (H3) are fulfilled.

P is $H^{-1/(k+1)}$ to L^2 locally solvable at $x_2 = 0$:

- $k = 0$, $\Sigma = \emptyset$;
- $k = 1$, (H4) is fulfilled;
- $k \geq 2$, (H5) is fulfilled.

Example 3.

In this example (H4) is not always satisfied. In $\mathbb{R}^3$, $x = (x_1, x_2, x_3)$, $\{x_1 = -1\} \subset \Omega$, $\Omega_\pm = \{x_1 \gtrless -1\}$, $f(x) = x_2 + x_2^3/3 - x_1 x_3$,

$$X_1(x,\xi) = \xi_1 - x_3\xi_3,\ X_2(x,\xi) = (1+x_1)\xi_3,\ X_0(x,\xi) = \xi_2 - x_1\xi_3,$$

$$X_3(x,\xi) = \sum_{j=0}^{2}\Big(\beta_j(x)X_j(x,\xi) + \gamma(x)\{X_0, X_j\}(x,\xi)\Big).$$

Then $\{X_1, X_0\} = -X_2$, $\{X_1, X_2\} = (2+x_1)\xi_3$, $\{X_2, X_0\} = 0$. Let

$$P = \sum_{j=1}^{2} X_j^* f X_j + X_3 + iX_0 + a_0.$$

(H1)–(H3) are all satisfied. If $x_0 = (-1, x_2^0, x_3^0)$,

$$\pi^{-1}(x_0) \cap \Sigma \neq \emptyset,\ \pi^{-1}(\Omega_\pm) \cap \Sigma = \emptyset.$$

We have

- if $x_0 \in f^{-1}(0) \cap \Omega_\pm$ then H^{-1} to L^2 local solvability near x_0;
- if $x_0 = (x_1^0 = -1, x_2^0, x_3^0) \in f^{-1}(0) \cap \Omega$ has fiber intersecting Σ then (H4) holds and $H^{-1/2}$ to L^2 local solvability near x_0.

The Schrödinger-type case.

Let $P_2 = \sum_{j=1}^{N} X_j^* f_j X_j + X_{N+1} + a_0$, with $f_1, \dots, f_N \in C^\infty(\Omega; \mathbb{R})$ and suppose:

(S1) $X_1, \dots, X_N$ have complex coefficients;

(S2) For all $x_0 \in \Omega$ $\exists$ cnnctd nghbrhd $V_{x_0} \subset \Omega$ and $g \in C^\infty(V_{x_0}; \mathbb{R})$ s.t.

- **(i)** $X_j g = 0$ on V_{x_0}, $1 \le j \le N$;
- **(ii)** $X_{N+1} g \neq 0$ on V_{x_0}.

Thm. (Federico-P.)

In the above hypotheses, P_2 is L^2 to L^2 locally solvable at any given $x_0 \in \Omega$.

No further regularity. Estimate $\operatorname{Im}(e^{\lambda g} P^* u, e^{\lambda g} u)$, $\lambda \gg 1$, $u \in C_0^\infty(K)$.

Example.

$\Omega_0 \subset \mathbb{R}_x^n \times \mathbb{R}_y$, $\Omega = \mathbb{R}_t \times \Omega_0$. Let $Q = Q(x)$ be a real-valued quadratic form,

$$X_j = D_{x_j} - i \frac{\partial Q}{\partial x_j}(x) D_y, \ \ 1 \le j \le n, \ X_{N+1} = D_t + Y,$$

where $Y = Y(x, y, D_x, D_y)$ is a first-order homogeneous diff. op. with real symbol. We may take $g = g(t) = t$. Then $P = \sum_{j=1}^{n} X_j^* f_j X_j + X_{N+1} + a_0$ is L^2 to L^2 locally solvable near each given $x_0 \in \Omega$.

The mixed-Schrödinger-type case.

Let $P_3 = \sum_{j=1}^{N} X_j^* f_j X_j + X_{N+1} + iX_0 + a_0$, with $f_1, \ldots, f_N \in C^\infty(\Omega; \mathbb{R})$ and suppose:

(M1) iX_0 has no critical point and $iX_0 f_j \geq 0$ on Ω, $1 \leq j \leq N$;

(M2) $\{X_0, X_j\} \equiv 0$ for all $1 \leq j \leq N$;

(M3) $\forall x_0 \in \Omega\ \exists K \subset \Omega$, $x_0 \in \mathring{K}$, $\exists C > 0$ s.t.

$$|\{X_0, X_{N+1}\}(x,\xi)|^2 \leq C\Big(X_0(x,\xi)^2 + \sum_{j=1}^{N} (iX_0 f_j(x)) X_j(x,\xi)^2\Big), \quad \forall (x,\xi) \in K \times \mathbb{R}^n.$$

Thm. (Federico)

P_3 is L^2 to L^2 locally solvable near any given $x_0 \in \Omega$.

Estimate $\mathrm{Re}(P^* u, -iX_0 u)$, $u \in C_0^\infty(K)$.
Main estimate (allows use of Fefferman-Phong and Poincaré)

$$2\mathrm{Re}(P^* u, -iX_0 u) \geq (P_0 u, u) + c_K \|X_0 u\|_0^2 - C_K \|u\|_0^2,$$

$$P_0 = X_0^2 + \sum_{j=1}^{N} [iX_0, X_j^* f_j X_j] - \epsilon [X_0, X_{N+1}]^* [X_0, X_{N+1}]$$

Final remarks.

(i) One has also "mildly" complex cases (need of control on subprincipal symbols).

(ii) How about local solvability in H^s spaces?
- E.g., when (H1)–(H4) hold, do we have H^s to $H^{s+1/2}$ solvability?
- What is the loss of derivatives?

(iii) How about propagation of singularities (equivalently Sobolev regularity) ($WF_s(u)$; real and complex cases) and hence semi-global (and global) solvability?

(iv) How about L^p to L^q local solvability ($p, q \neq 2$, start with $1 < p, q < \infty$)?