Sufficient Condition for Strongly Starlikeness of Normalized Mittag-Leffler function

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Abstract

In the present investigation a sufficient condition is obtained for normalized Mittag-Leffler function to be starlike and strongly starlike in the open unit disk. Results obtained are new and their usefulness are depicted by deducing several interesting corollaries and examples.

Introduction

Let $N$ denote the class of analytic functions in the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ and $A$ denote the subclass of $N$, which are normalized by the condition $f(0) = 0 = f'(0) - 1$ and have representation of the form

$$f(z) = z + \sum_{n \geq 2} a_n z^n, \quad z \in D.$$  \hspace{1cm} (1)

A function $f$ is said to be univalent in a domain $D$ if it is one-to-one in $D$. A function $f \in A$ is called starlike, denoted by $f \in S^*$, if it is univalent in $D$ and $f(D)$ is a starlike domain with respect to the origin. For a given $0 \leq \alpha < 1$, a function $f \in A$ is called starlike function of order $\alpha$, class of such function denoted by $S^\alpha$, if and only if $\Re(f'(z)/f(z)) > \alpha$, $z \in D$. Further, a function $f \in A$ is called convex function, if $1 + \Re(f''(z)/f'(z)) > 0$, $z \in D$. It is well known that $S^0(0) = S^0$. Further, let $S^\alpha(0), 0 < \alpha \leq 1$, be the class of strongly starlike functions of order defined by

$$S^\alpha(0) = \left\{ f \in A : \left| \frac{f'(z)}{f(z)} \right| < \frac{\alpha}{1 - \alpha}, \quad z \in D \right\}.$$  \hspace{1cm} (2)

Note that $S^1(0) \equiv S^0$. For more details about these classes can be found in [7].

If $f \in H$, then the function $f$ is said to be subordinate to $p$, written as $f(z) \prec p(z)$ ($z \in D$), if there exists a Schwarz function $w$ with $w(0) = 0$ and $|w(z)| < 1$ ($z \in D$) such that $f(z) = w(p(z))$. In particular, if $q$ is univalent in $D$, then we have the following equivalence:

$$f(z) \prec q(z) \iff f(0) = q(0) \quad \text{and} \quad f(D) \subset q(D).$$

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Normalized Mittag-Leffler function

The function

$$E_{\lambda}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\lambda n + 1)} \quad (\lambda \in \mathbb{C}, \Re(\lambda) > 0, \quad z \in \mathbb{C}),$$  \hspace{1cm} (3)

was introduced by Mittag-Leffler in 1902 [7] in connection with the summation of divergent series. An important generalization

$$E_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\lambda n + \mu)} \quad (\lambda, \mu \in \mathbb{C}, \Re(\lambda) > 0, \Re(\mu) > 0, \quad z \in \mathbb{C}),$$  \hspace{1cm} (4)

was introduced by Wiman [7, 7]. The Mittag-Leffler function has a close connection to differential equations of fractional order and integral equations of Abel type, such equations becoming more and more popular in modelling natural and technical processes.

Observ. That Mittag-Leffler function $E_{\lambda}(z)$ does not belong to the family $A$. Thus, it is natural to consider the following normalization of Mittag-Leffler function $E_{\lambda,\mu}(z)$. If $f \in A$, then

$$S^\alpha(0) = \left\{ f \in A : \left| \frac{f'(z)}{f(z)} \right| < \frac{\alpha}{1 - \alpha}, \quad z \in D \right\}.$$  \hspace{1cm} (2)

Whilst formula (5) holds for complex-valued $\lambda, \mu$, however in this paper we shall restrict our attention to the case of real-valued $\lambda, \mu$. It is easy to see that $E_{\lambda,\mu}(z)$ satisfy the following relation:

$$E_{\lambda,\mu}(z) = \left( \frac{\mu}{\lambda} \right)^{1/\alpha} E_{\lambda,\mu}(\left( \frac{\mu}{\lambda} \right)^{1/\alpha} z).$$  \hspace{1cm} (6)

Several researchers studied families of analytic functions involving special functions to find conditions such that the image of functions under $H$ have certain geometric properties like univalency, starlikeness or convexity. In this context many results are available in the literature regarding the hypergeometric functions [7], Bessel functions [7, 7], Wright function [7] and Mittag-Leffler functions [7]. In the present paper, we study some properties of Normalized Mittag-Leffler function $E_{\lambda,\mu}(z)$.

Necessary Lemmas

To prove results in this section, we shall need following Lemma:

**Lemma 0.1.** (Halkoer and Ruchewich [7]). Let $G(z)$ be convex and univalent in $D$ with $G(0) = 1$. Let $F(z)$ be analytic in $D$. $F(0) = 1$ and $F(z) \subset G(z)$ in $D$. Then for all $n \in N \cup \{0\}$, we have

$$(n+1)! \int_0^1 e^{t} F(t) dt < (n+1)! \int_0^1 e^{t} G(t) dt.$$  \hspace{1cm} (7)

Further it is easy to see that, under the hypothesis $\lambda \geq 1$, the inequality $\Gamma(\alpha + \mu) \leq \Gamma(\mu + 1)$, holds for all $n \in N$, which is equivalent to

$$\sum_{n=0}^{\infty} \frac{\mu^n}{\Gamma(\lambda n + \mu)} \leq \frac{\mu}{\lambda} \sum_{n=0}^{\infty} \frac{\mu^n}{\Gamma(\lambda n + \mu)}.$$  \hspace{1cm} (8)

**Lemma 0.2.** (Fujii). If $a_0 \geq 0$, $\{a_n\}$ and $\{a_n - (n + 1)a_{n+1}\}$ both are nonincreasing, then the function $f$ defined by (1) is in $S^\alpha$.

**Theorem 0.1.** If $\lambda \geq 1$ and $\mu \geq \lambda$, then $E_{\lambda,\mu}(z) \in S^\alpha(0)$, where

$$\alpha = \frac{2}{\pi} \arcsin \left( \frac{\lambda}{\sqrt{\lambda^2 - 1}} \right).$$  \hspace{1cm} (9)

**Proof.** Using the inequality (7), we get

$$\left| E_{\lambda,\mu}(z) - 1 \right| \leq \sum_{n=0}^{\infty} \frac{\mu^n}{\Gamma(\lambda n + \mu)} \left( \frac{\mu}{\lambda} \right)^{1/\alpha} \left( \frac{\mu}{\lambda} \right)^{1/\alpha} z.$$  \hspace{1cm} (10)

Putting $\alpha = 1$ in Corollary 0.1, we get

$$\mu - 1 \Rightarrow \lambda \geq 1, \quad \mu \geq \lambda.$$  \hspace{1cm} (11)

Now from (11) and (12), we conclude that

$$\arg \left( E_{\lambda,\mu}(z) \right) \leq \arg \left( E_{\lambda,\mu}(z) \right) + \arg \left( E_{\lambda,\mu}(z) \right) \leq \frac{\pi}{2}.$$  \hspace{1cm} (12)

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In order to prove $E_{\lambda,\mu}(\alpha)(z)$ is a convex function in $D$. It is sufficient to prove that $zE_{\lambda,\mu}(\alpha)(z)$ is starlike function in $D$. Let

$$zE_{\lambda,\mu}^{(n)}(\alpha)(z) = z \sum_{n=1}^{\infty} \frac{a_n}{\Gamma(n(\alpha - 1) + \mu)} z^n = \sum_{n=1}^{\infty} A_n z^n$$

where $A_1 = 1$ and for $n \geq 2$,

$$A_n = \frac{a_n}{\Gamma(n(\alpha - 1) + \mu)}$$

In view of Lemma 0.2, it is sufficient to prove that $(nA_n)$ and $(nA_n - (n - 1)A_{n+1})$ are nonincreasing sequences for all $n \geq 1$.

Now

$$nA_n - (n+1)A_{n+1} = \frac{n \Gamma(\alpha - 1)}{\Gamma(n(\alpha - 1) + \mu)} (\phi(n) - \psi(n)).$$

where

$$\phi(n) = \frac{\Gamma(n(\alpha + \mu))}{\Gamma(n(\alpha - 1) + \mu)} > 0 \text{ and } \psi(n) = \frac{(n+1)^2}{n^2} > 0 \text{ (for all } n \geq 1)$$

Now $\phi(n)$ is a strictly increasing function of $n$ for all $n \geq 1$. Similarly $\psi(n)$ is a strictly decreasing function of $n$. In order to prove that $nA_n - (n+1)A_{n+1} \geq 0$ for all $n \geq 1$, it sufficient to show that $\phi(1) \geq \psi(1)$ i.e. $\Gamma(\lambda + \mu) \geq \Gamma(\mu)$, which is true in view of hypothesis of Theorem. Further

$$n(\phi(n) - \psi(n)) = \frac{n^2 \Gamma(\mu + 1)}{\Gamma(n(\alpha - 1) + \mu)(\alpha + 1)} \left[ 2(\lambda n^2) + 2(n+1)^2 \right] \geq 0 \text{ (for all } n \geq 1).$$

Taking the difference of first two term on right hand side and using the same reasoning as used above, we get $n(\phi(n) - \psi(n)) \geq 0$ for all $n \geq 1$.

References