Sufficient Condition for Strongly Starlikeness of Normalized Mittag-Leffler function

Deepak Bansal

University College of Engineering and Technology, Bikaner, Rajasthan, India

Department of HEAS (Mathematics), Bikaner Technical University.

Contact Information:
Department of Mathematics
Bikaner Technical University
Bikaner, Rajasthan, India

Phone: +91 9413118022
Email: dr.deepakbansal79@gmail.com

Abstract

In the present investigation a sufficient condition is obtained for normalized Mittag-Leffler function to be starlike and strongly starlike in the open unit disk. Results obtained are new and their usefulness are depicted by deducing several interesting corollaries and examples.

Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

and $\mathcal{A}$ denote the subclass of $\mathcal{H}$, which are normalized by the condition $f(0) = 0 = f'(0) - 1$ and have representation of the form

$$f(z) = z + \sum_{n\geq 2} a_n z^n, \quad z \in \mathbb{D}. \tag{1}$$

A function f is said to be univalent in a domain $\mathbb{D}$ if it is one-to-one in $\mathbb{D}$. A function $f \in \mathcal{A}$ is called starlike, denoted by $f \in \mathcal{S}^*$ if f is univalent in $\mathbb{D}$ and $f(\mathbb{D})$ is a starlike domain with respect to the origin. For a given $0 \leq \alpha < 1$, a function $f \in \mathcal{A}$ is called starlike function of order α, class of such function denoted by $\mathcal{S}^*(\alpha)$, if and only if $\Re\left(zf'(z)/f(z)\right) > \alpha$, $z \in \mathbb{D}$. Further, a function $f \in \mathcal{A}$ is called a convex function, if $1 + \Re\left(zf''(z)/f'(z)\right) > 0$, $z \in \mathbb{D}$. It is well known that $\mathcal{S}^*(0) = \mathcal{S}^*$. Further, let $\widetilde{\mathcal{S}}^*(\alpha)$, $0 < \alpha \leq 1$, be the class of strongly starlike functions of order α defined by

$$\widetilde{\mathcal{S}}^*(\alpha) = \left\{ f \in \mathcal{A} : \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D} \right\}. \tag{2}$$

Note that $\widetilde{\mathcal{S}}^*(1) \equiv \mathcal{S}^*$. For more details about these classes can be found in [?].

If $f, g \in \mathcal{H}$, then the function f is said to be subordinate to g, written as $f(z) \prec g(z)$ $(z \in \mathbb{D})$, if there exists a Schwarz function $w \in \mathcal{H}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{D})$ such that $f(z) = g(w(z))$. In particular, if g is univalent in $\mathbb{D}$, then we have the following equivalence:

$$f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{D}) \subset g(\mathbb{D}).$$

Contents

1. Normalized Mitteg-Leffler function.
2. Necessary Lemmas.
3. Results.
4. References.

Normalized Mittag-Leffler function

The function

$$E_\lambda(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\lambda n + 1)} \quad (\lambda \in \mathbb{C}, \Re(\lambda) > 0, z \in \mathbb{C}), \tag{3}$$

was introduced by Mittag-Leffler in 1902 [?] in connection with the summing of divergent series. An important generalization

$$E_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\lambda n + \mu)} \quad (\lambda, \mu \in \mathbb{C}, \Re(\lambda) > 0, \Re(\mu) > 0, z \in \mathbb{C}), \tag{4}$$

was introduced by Wiman [?, ?]. The Mittag-Leffler function has a close connection to differential equations of fractional order and integral equations of Abel type, such equations becoming more and more popular in modelling natural and technical process.

Observe that, Mittag-Leffler function $E_{\lambda,\mu}$ does not belong to the family $\mathcal{A}$. Thus, it is natural to consider the following normalization of Mittag-Leffler function in $\mathbb{D}$:

$$\begin{aligned} \mathbb{E}_{\lambda,\mu}(z) &= \Gamma(\mu)\, z E_{\lambda,\mu}(z) \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\lambda n + \mu)} z^{n+1} \; (\lambda, \mu \in \mathbb{C}; \Re(\lambda) > 0, \mu \neq 0, -1, \cdots z \in \mathbb{D}). \end{aligned} \tag{5}$$

Whilst formula (5) holds for complex-valued λ, μ, however in this paper we shall restrict our attention to the case of real-valued λ, μ. It is easy to see that $\mathbb{E}_{\lambda,\mu}$ satisfy the following relation:

$$\lambda z \mathbb{E}'_{\lambda,\mu}(z) = (\mu - 1)\mathbb{E}_{\lambda,\mu-1}(z) + (\lambda - \mu + 1)\,\mathbb{E}_{\lambda,\mu}(z) \tag{6}$$

Several researchers studied families of analytic functions involving special functions to find conditions such that the image of functions under $\mathbb{D}$ have certain geometric properties like univalency, starlikeness or convexity. In this concern many results are available in the literature regarding the hypergeometric functions [?], Bessel functions [?, ?, ?], Wright function [?] and Mittag-Leffler functions [?]. In the present paper, we study some properties of Normalized Mittag-Leffler function $\mathbb{E}_{\lambda,\mu}$.

Necessary Lemmas

To prove results in this section, we shall need following Lemma:

Lemma 0.1. (Halenbeck and Ruscheweyh [?]). *Let $G(z)$ be convex and univalent in $\mathbb{D}$ with $G(0) = 1$. Let $F(z)$ be analytic in $\mathbb{D}$, $F(0) = 1$ and $F(z) \prec G(z)$ in $\mathbb{D}$. Then for all $n \in \mathbb{N} \cup \{0\}$, we have*

$$(n+1)z^{-n-1}\int_0^z t^n F(t)\,dt \prec (n+1)z^{-n-1}\int_0^z t^n G(t)\,dt.$$

Further it is easy to see that, under the hypothesis $\lambda \geq 1$, the inequality $\Gamma(n + \mu) \leq \Gamma(\lambda n + \mu)$, holds for all $n \in \mathbb{N}$, which is equivalent to

$$\frac{\Gamma(\mu)}{\Gamma(\lambda n + \mu)} \leq \frac{1}{\mu(\mu+1)\cdots(\mu+n-1)}, \quad \forall n \in \mathbb{N}. \tag{7}$$

Lemma 0.2. (Féjer). *If $a_n \geq 0$, $\{na_n\}$ and $\{na_n - (n+1)a_{n+1}\}$ both are nonincreasing, then the function f defined by* (1) *is in $\mathcal{S}^*$.*

Results

Theorem 0.1. *If $\lambda \geq 1$ and $\mu \geq \frac{3+\sqrt{17}}{2}$, then $\mathbb{E}_{\lambda,\mu} \in \widetilde{\mathcal{S}}^*(\alpha)$, where*

$$\alpha = \frac{2}{\pi}\arcsin\left(\eta\sqrt{1 - \frac{\eta^2}{4}} + \frac{\eta}{2}\sqrt{1 - \eta^2}\right), \tag{8}$$

for $\eta = (3\mu + 2)/\mu^2$.

Proof. Using the inequality (7), we get

$$\begin{aligned} \left|\mathbb{E}'_{\lambda,\mu}(z) - 1\right| &\leq \sum_{n=1}^{\infty} \frac{(n+1)\,\Gamma(\mu)}{\Gamma(\lambda n + \mu)}|z|^n \\ &\leq \sum_{n=1}^{\infty} \frac{n+1}{\mu(\mu+1)\cdots(\mu+n-1)} \\ &= \sum_{n=1}^{\infty} \frac{n}{\mu(\mu+1)\cdots(\mu+n-1)} + \sum_{n=1}^{\infty} \frac{1}{\mu(\mu+1)\cdots(\mu+n-1)} \\ &< \frac{1}{\mu} + \frac{1}{\mu}\left\{1 + \frac{1}{\mu+1} + \left(\frac{1}{\mu+1}\right)^2 + \cdots\right\} + \frac{1}{\mu}\left\{1 + \frac{1}{\mu+1} + \left(\frac{1}{\mu+1}\right)^2 + \cdots\right\} \\ &= \frac{(3\mu+2)}{\mu^2} = \eta. \end{aligned} \tag{9}$$

Note that, under the hypothesis, $0 < \eta \leq 1$. From (9), we conclude that

$$\mathbb{E}'_{\lambda,\mu}(z) \prec 1 + \eta z \quad z \in \mathbb{D}.$$

which gives

$$|\arg(\mathbb{E}'_{\lambda,\mu}(z))| < \arcsin\eta, \quad z \in \mathbb{D}. \tag{11}$$

Using Lemma 0.1, for $F(z) = \mathbb{E}'_{\lambda,\mu}(z)$ and $G(z) = 1 + \eta z$ with $n = 0$, we get

$$\frac{\mathbb{E}_{\lambda,\mu}(z)}{z} \prec 1 + \frac{\eta}{2}z, \quad z \in \mathbb{D},$$

consequently

$$\left|\arg\left(\frac{\mathbb{E}_{\lambda,\mu}(z)}{z}\right)\right| < \arcsin\frac{\eta}{2}, \quad z \in \mathbb{D}. \tag{12}$$

Now from (11) and (12), we conclude that

$$\begin{aligned} \left|\arg\left(\frac{z\mathbb{E}'_{\lambda,\mu}(z)}{\mathbb{E}_{\lambda,\mu}(z)}\right)\right| &= \left|\arg\left(\frac{z}{\mathbb{E}_{\lambda,\mu}(z)}\right) + \arg\left(\mathbb{E}'_{\lambda,\mu}(z)\right)\right| \\ &\leq \left|\arg\left(\frac{z}{\mathbb{E}_{\lambda,\mu}(z)}\right)\right| + \left|\arg\left(\mathbb{E}'_{\lambda,\mu}(z)\right)\right| \\ &< \arcsin\frac{\eta}{2} + \arcsin\eta \\ &= \arcsin\left(\eta\sqrt{1 - \frac{\eta^2}{4}} + \frac{\eta}{2}\sqrt{1 - \eta^2}\right) \end{aligned}$$

i. e. $\mathbb{E}_{\lambda,\mu}(z) \in \widetilde{\mathcal{S}}^*(\alpha)$, for α given in (8). □

Corollary 0.1. *Let $\lambda \geq 1$ and $\mu \geq \frac{3+\sqrt{17}}{2}$. If $0 < \alpha \leq 1$ and*

$$\eta = \frac{(3\mu+2)}{\mu^2} = 2\nu\sqrt{\frac{5 - 4\sqrt{1-\nu^2}}{16\nu^2 + 9}}, \tag{13}$$

where $\nu = \sin\left(\frac{\alpha\pi}{2}\right)$. Then $\mathbb{E}_{\lambda,\mu} \in \widetilde{\mathcal{S}}^*(\alpha)$.

Proof. If we put η from (13) to (8), we obtain α. □

Putting $\alpha = 1$ in Corollary 0.1, we get

$$\nu = 1 \Rightarrow \eta = \frac{3\mu+2}{\mu^2} = \frac{2}{\sqrt{5}}.$$

Corollary 0.2. *If $\lambda \geq 1$ and $\mu = \mu^*$, where μ^* is positive root of*

$$2\mu^2 - 3\sqrt{5}\mu - 2\sqrt{5} = 0, \tag{10}$$

then $\mathbb{E}_{\lambda,\mu}(z) \in \mathcal{S}^*$.

If $\lambda = 1$ and $\mu = 4$ then $\eta = \frac{3\mu+2}{\mu^2} = 7/8$. From (8), we get

$$\alpha = \frac{2}{\pi}\arcsin\left(\frac{7}{128}[\sqrt{207} + \sqrt{15}]\right) \tag{14}$$

and thus from Theorem 0.1, we have

$$\mathbb{E}_{1,4}(z) = \frac{6(e^z - 1 - z) - 3z^2}{z^2} \in \widetilde{\mathcal{S}}^*(\alpha). \tag{15}$$

Theorem 0.2. *If $\lambda > 0$, $\beta > 0$ and $\Gamma(\lambda + \mu) \geq 8\,\Gamma(\mu)$, then the function $\mathbb{E}_{\lambda,\mu}(z)$ is convex in $\mathbb{D}$.*

Proof. In order to prove $\mathbb{E}_{\lambda,\mu}(z)$ is a convex function in $\mathbb{D}$. It is sufficient to prove that $z\mathbb{E}'_{\lambda,\mu}(z)$ is starlike function in $\mathbb{D}$. Let

$$z\mathbb{E}'_{\lambda,\mu}(z) = z + \sum_{n=2}^{\infty} \frac{n\Gamma(\beta)}{\Gamma(\lambda(n-1)+\mu)} z^n = \sum_{n=1}^{\infty} A_n z^n. \quad (16)$$

where $A_1 = 1$ and for $n \geq 2$,

$$A_n = \frac{n\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)}.$$

In view of Lemma 0.2, it is sufficient to prove that $\{nA_n\}$ and $\{nA_n - (n+1)A_{n+1}\}$ are nonincreasing sequences for all $n \geq 1$. Now

$$na_n - (n+1)a_{n+1} = \frac{n^2\Gamma(\mu)}{\Gamma(\lambda n+\mu)} \left[\phi(n) - \psi(n)\right].$$

where

$$\phi(n) = \frac{\Gamma(\lambda n+\mu)}{\Gamma(\lambda(n-1)+\mu)} > 0 \;\; and \;\; \psi(n) = \frac{(n+1)^2}{n^2} > 0 \; (for \; all \; n \geq 1)$$

Now $\phi(n)$ is a strictly increasing function of n for all $n \geq 1$. Similarly $\psi(n)$ is a strictly decreasing function of n. In order to prove that $nA_n - (n+1)A_{n+1} \geq 0$ for all $n \geq 1$, it sufficient to show that $\phi(1) \geq \psi(1)$ i.e. $\Gamma(\lambda+\mu) \geq 4\Gamma(\mu)$, which is true in view of hypothesis of Theorem. Further

$$na_n - 2(n+1)a_{n+1} + (n+2)a_{n+2} = \frac{n^2\Gamma(\mu)}{\Gamma[\lambda(n-1)+\mu]} - \frac{2(n+1)^2\Gamma(\mu)}{\Gamma[\lambda n+\mu]} + \frac{(n+2)^2\Gamma(\mu)}{\Gamma[\lambda(n+1)+\mu]}$$

Taking the difference of first two term on right hand side and using the same reasoning as used above, we get $na_n - 2(n+1)a_{n+1} + (n+2)a_{n+2} \geq 0$ for all $n \geq 1$. □

References

- D. Bansal and J. K. Prajapat, Certain geometric properties of the Mittag-Leffler functions, Complex Var. Elliptic Eq. **61(3)** (2016), 338–350.
- P. L. Duren, Univalent Functions, Grundlehren der Mathematischen Wissenschaften, Band 259, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, 1983.
- M. M. Dzherbashyan, Integral Transforms and Representations of Functions in the Complex Plane, Nauka, Moscow, 1966 (in Russian).
- R. Gorenflo and S. Vessella, Abel Integral Equations. Analysis and Applications, Springer-Verlag, Berlin, Heidelberg, New York and Tokyo, 1991.
- D. J. Halenbeck and S. Ruscheweyh, Subordinations by convex functions, Proc. Amer. Math. Soc. **52** (1975), 191–195.
- S. S. Miller and P. T. Mocanu, Differential Subordinations, Theory and Applications, New York-Basel, Marcel Dekker, 2000.
- S. S. Miller and P. T. Mocanu, Univalence of Gaussian and confluent hypergeometric functions, Proc. Amer. Math. Soc. **110(2)** (1990), 333–342.
- G. M. Mittag-Leffler, Sur la nouvelle fonction $E_\alpha(x)$, C.R. Acad. Sci. Paris **137** (1903), 554–558.
- A. Wiman, Über den Fundamental satz in der Theorie der Funcktionen, $E_\alpha(x)$, Acta Mathematica, **29** (1905), 191–201.
- A. Wiman, Über die Nullstellun der Funktionen $E_\alpha(x)$, Acta Mathematica **29** (1905a), 217–234.