Abstract

We study a class of anharmonic oscillators within the framework of the Weyl-Hörmander calculus. By associating a Hörmander metric to a given anharmonic oscillator we extend the so-called Shubin classes associated to the harmonic oscillator and the corresponding pseudo-differential calculus. Spectral properties of negative powers of anharmonic oscillators, as well as of the operator itself, are derived.

Introduction

In the study of the Schrödinger equation $i\partial_t \psi = -\Delta + V(x)\psi$, the analysis of the energy levels is often reduced to the corresponding eigenvalue problem for the operator $-\Delta + V(x)$. Spectral properties of the anharmonic oscillator, on $\mathbb{R}$ or more generally on $\mathbb{R}^n$, with different potentials $V$ have been studied (c.f. [1],[2],[4]) by several authors in the last 40 years. However, the exact solution of the eigenvalue problem is still unknown. Here we consider a more general case on $\mathbb{R}^n$ where $\psi$ is a prototype of the form

\[ A = (-\Delta)^{k/2} + |x|^{\alpha}, \quad k \geq 1 \text{ integer}, \quad \alpha \geq 0, \]

and more generally we consider the form

\[ T = V(D) + p(x), \]

where $p, q$ are special polynomials on $\mathbb{R}^n$. In particular we write $p \in \mathcal{P}_n$, $q \in \mathcal{P}_m$, for some integers $k, l \geq 1$, where we define

\[ \mathcal{P}_n = \left\{ p: \mathbb{R}^n \to \mathbb{R}, \text{ with } \deg(p) = 2k \right\}. \]

Therefore, for $p \in \mathcal{P}_n$ and (similarly for $q$), there exists $p_0 > 0$ such that $p(x) > p_0 > 0$, for every $x \in \mathbb{R}^n$.

Weyl-Hörmander classes associated to the anharmonic oscillators

1. In the case of the general anharmonic oscillator $T$ in (2) with (rescaled) symbol $r$ we have

\[ r(x, \xi) = p(x) + q(x) = \mathcal{S}(m^{2k}, m^{\alpha}), \]

where the Hörmander metric $m^\alpha$ and the weight metric $m^{2k}$ are given by

\[ m^\alpha(x, \xi) = \frac{1}{\sqrt{\pi^k + |\xi|^\alpha}}, \quad \mathcal{S}(m^{2k}, m^{\alpha}) \]

and

\[ m^{2k}(x, \xi) = \frac{1}{\sqrt{\pi^k + |\xi|^\alpha}}, \quad \mathcal{S}(m^{2k}, m^{\alpha}) \]

2. In the case of the prototype of the anharmonic oscillator $A$ as in (1) the metric is equivalent to the metric

\[ \frac{dx^2}{\pi^k + |x|^\alpha + |\xi|^{\alpha}}, \quad \mathcal{S}(m^{2k}, m^{\alpha}) \]

3. In the symmetric case where $k = l$ in (1) the metric associated to the operator $A$ is equivalent to

\[ \frac{dx^2}{\pi^k + |x|^\alpha + |\xi|^{\alpha}}, \quad \mathcal{S}(m^{2k}, m^{\alpha}) \]

which corresponds to the symplectic metric defining the Shubin classes associated to the harmonic oscillator.

Associated symbol classes and operators

Let $a \in \mathbb{R}^d$. We say that the function $a \in C^0(\mathbb{R}^d)$ is in the class of symbols $\Sigma^m_{\mathcal{O}}(\mathbb{R}^d)$, if

\[ |\partial^\alpha_p \partial^\beta_q a(x, \xi)| \leq C_{\alpha\beta} \langle m(x, \xi) \rangle^{\alpha + \beta - |\alpha| - \beta} \text{ for all } \alpha, \beta \in \mathbb{N}^d, \]

where we have denoted $m(x, \xi) = (1 + |x|^k + |\xi|^l)^{\alpha}$. The associated operators are denoted by $\mathcal{O}^{\Sigma^m_{\mathcal{O}}}(\mathbb{R}^d)$, and in particular

\[ \Psi^{\Sigma^m_{\mathcal{O}}}(\mathbb{R}^d) = \mathcal{O}^{\Sigma^m_{\mathcal{O}}}(\mathbb{R}^d). \]

For example the prototype anharmonic oscillator $A$ as in (1) is an operator in $\Psi^{\Sigma^m_{\mathcal{O}}}(\mathbb{R}^d)$.

Pseudo-differential calculus on $\Sigma^m_{\mathcal{O}}(\mathbb{R}^d)$

The following can be viewed as a consequence of the Weyl-Hörmander calculus:

1. The class of operators $\mathcal{O}^{\Sigma^m_{\mathcal{O}}}(\mathbb{R}^d)$ forms an algebra of operators that is stable under the adjoint.
2. Moreover, we have the asymptotic formula

\[ \sum_{\alpha + \beta - |\alpha| - \beta} \frac{m(x, \xi)}{\alpha + \beta - |\alpha| - \beta} \]

3. The operators in $\Psi^{\Sigma^m_{\mathcal{O}}}(\mathbb{R}^d)$ extend boundary to $L^2(\mathbb{R}^d)$. Furthermore, there exists $C > 0$ and $\alpha, \beta \in \mathbb{N}$ such that $A \in \mathcal{O}^{\Sigma^m_{\mathcal{O}}}(\mathbb{R}^d)$, then

\[ |A(x, \xi)| \leq C \langle m(x, \xi) \rangle^\alpha. \]

Associated Sobolev spaces

Using the functional calculus on the (compact, positive operator) $A$ as in (1), we define the operator $A^s$, for $s \in \mathbb{R}$, by

\[ A^s = \sum_{\alpha + |\alpha| - |\alpha| - |\alpha|} \frac{m(x, \xi)}{\alpha + \beta - |\alpha| - \beta} \]

and

\[ \mathcal{O}^{\Sigma^m_{\mathcal{O}}}(\mathbb{R}^d) \]

The Sobolev spaces related to $A$, denoted by $A^{s \mathcal{O}}(\mathbb{R}^d)$, for $s \in \mathbb{R}$, $k \geq 1$ integers, is the subspace $\mathcal{S}(\mathbb{R}^d)$ of the completion of $\mathcal{O}^{\Sigma^m_{\mathcal{O}}}(\mathbb{R}^d)$ for the norm

\[ \|A^s - A\|_{\mathcal{O}^{\Sigma^m_{\mathcal{O}}}(\mathbb{R}^d)}. \]

Continuity Properties

The identification of the Sobolev spaces $\mathcal{S}^{m_{\mathcal{O}}}(\mathbb{R}^d)$ with suitable Sobolev spaces $H^m, m \geq 1$, in the Weyl-Hörmander setting, and the general theory yield

1. Let $m \in \mathbb{R}$ and $l \geq 1$ integers. If $a \in \mathcal{S}^{m_{\mathcal{O}}}(\mathbb{R}^d)$, then

\[ \mathcal{O}^{\Sigma^m_{\mathcal{O}}}(\mathbb{R}^d) \]

2. More generally, equivalence of quantizations in our particular case yield

\[ \text{For } a \in \mathcal{S}^{m_{\mathcal{O}}}(\mathbb{R}^d), \text{ we have} \mathcal{O}^{\Sigma^m_{\mathcal{O}}}(\mathbb{R}^d) \]

References


