On a class of anharmonic oscillators

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Abstact

We study a class of anharmonic oscillators within the framework of the Weyl-Hörmander calculus. By associating a Hörmander metric to a given anharmonic oscillator we extend the so-called *Shubin classes* associated to the harmonic oscillator and the corresponding pseudo-differential calculus. Spectral properties of negative powers of anharmonic oscillators, as well as of the operator itself, are derived.

Introduction

In the study of the **Schrödinger equation** $i\partial_t\psi = -\Delta\psi + V(X)\psi$ the analysis of the energy levels is often reduced to the corresponding eigenvalue problem for the operator $-\Delta + V(x)$.
Spectral properties of the anharmonic oscillator, on $\mathbb{R}$ or more generally on $\mathbb{R}^n$, with different potentials V have been studied (c.f. [1],[2],[4]) by several authors in the last 40 years. However, the exact solution of the eigenvalue problem is still unknown.
Here we consider a more general case on $\mathbb{R}^n$ where a **prototype** is of the form

$$\mathcal{A} = (-\Delta)^l + |x|^{2k} , \quad \text{where} \quad k, l \geq 1 \quad \text{integers} , \tag{1}$$

and **more generally** we consider operators of the form

$$T = q(D) + p(x) , \tag{2}$$

where p, q are special polynomials on $\mathbb{R}^n$. In particular we write $p \in \mathcal{P}_{2k}, q \in \mathcal{P}_{2l}$, for some integers $k, l \geq 1$, where we have defined

$$\mathcal{P}_{2k} = \left\{ p : \mathbb{R}^n \to \mathbb{R}, \text{with } \deg(p) = 2k \,, \text{and } \lim_{|x|\to\infty} \inf \frac{p(x)}{|x|^{2k}} > 0 \right\} .$$

Therefore, for $p \in \mathcal{P}_{2k}$ (and similarly for q), there exists $p_0 > 0$ such that

$$p(x) + p_0 > 0 \,, \quad \text{for every} \quad x \in \mathbb{R}^n .$$

Weyl-Hörmander classes associated to the anharmonic oscillators

- **In the case of the general anharmonic oscillator** T as in (2) with (rescalled) symbol τ we have

$$\tau(x, \xi) = p(x) + q(\xi) \in S(M^{p,q}, g^{p,g}) ,$$

  where the Hörmander metric $g^{p,q}$, and the g-weight $M^{p,q}$ are given by

$$g^{p,q}_{x,\xi}(dx, d\xi) = \frac{dx^2}{\underbrace{(p_0 + q_0 + p(x) + q(\xi))}_{>0}{}^{\frac{1}{k}}} + \frac{d\xi^2}{(p_0 + q_0 + p(x) + q(\xi))^{\frac{1}{l}}} , \tag{3}$$

  and

$$M^{p,q}(x, \xi) = p_0 + q_0 + p(x) + q(\xi) \tag{4}$$

- **In the case of the prototype of the anharmonic oscillator** $\mathcal{A}$ as in (1) the metric (3) is equivalent to the metric

$$g^{k,l}_{x,\xi}(dx, d\xi) = \frac{dx^2}{(1 + |x|^{2k} + |\xi|^{2l})^{\frac{1}{k}}} + \frac{d\xi^2}{(1 + |x|^{2k} + |\xi|^{2l})^{\frac{1}{l}}} . \tag{5}$$

- **In the symmetric case where** $k = l$ in (1) the metric associated to the operator $\mathcal{A}$ is equivalent to

$$g_{x,\xi}(dx, d\xi) = \frac{dx^2}{1 + |x|^2 + |\xi|^2} + \frac{d\xi^2}{1 + |x|^2 + |\xi|^2} ,$$

which corresponds to the symplectic metric defining the *Shubin classes* associated to the *harmonic oscillator*.

Associated symbol classes and operators

Let $m \in \mathbb{R}$. We say that the function $a \in C^\infty(\mathbb{R}^{2n})$ is in the **class of symbols** $\Sigma^m_{k,l}(\mathbb{R}^n)$, if

$$|\partial^\alpha_x \partial^\beta_\xi a(x, \xi)| \leq C_{\alpha,\beta} \Lambda(x, \xi)^{m - \frac{|\alpha|}{k} - \frac{|\beta|}{l}} , \quad \text{for all} \quad \alpha, \beta \in \mathbb{N}^n ,$$

where we have denoted $\Lambda(x, \xi) := (1 + |x|^{2k} + |\xi|^{2l})^{\frac{1}{2}}$. for $k, l \geq 1$ integers. The **associated operators** are denoted by $\Psi\Sigma^m_{k,l}(\mathbb{R}^n)$, and in particular

$$\Psi\Sigma^m_{k,l}(\mathbb{R}^n) = Op^w(\Sigma^m_{k,l}(\mathbb{R}^n)) .$$

For example the **prototype anharmonic oscillator** $\mathcal{A}$ as in (1) is an operator in $\Psi\Sigma^2_{k,l}(\mathbb{R}^n)$.

Pseudo-differential calculus on $\Sigma^m_{k,l}(\mathbb{R}^n)$

The following can be viewed as a consequence of the Weyl-Hörmander calculus:

- The class of operators $\cup_{m\in\mathbb{R}} \Psi\Sigma^m_{k,l}(\mathbb{R}^n)$ forms an algebra of operators that is stable under taking the adjoint.
- Let $m_1, m_2 \in \mathbb{R}$ and let $k, l \geq 1$ integers. If $a \in \Sigma^{m_1}_{k,l}$ and $b \in \Sigma^{m_2}_{k,l}$, then there exists $c \in \Sigma^{m_1+m_2}_{k,l}$ such that $Op^w(a) \circ Op^w(b) = Op^w(c)$. Moreover, we have the asymptotic formula

$$c \sim \sum_\alpha \frac{(2\pi i)^{-|\alpha|}}{\alpha!} (\partial^\alpha_\xi a)(\partial^\alpha_x b) .$$

- The operators in $\Psi\Sigma^0_{k,l}$ extend boundedly to $L^2(\mathbb{R}^n)$. Furthermore, there exists $C > 0$ and $N \in \mathbb{N}$ such that if $A = Op^w(a) \in \Psi\Sigma^0_{k,l}$, then

$$\|A\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C\|a\|_{\Sigma^0_{k,l};N} ,$$

  where $\| \cdot \|_{\Sigma^0_{k,l};N}$, denotes the inherited seminorm in the class in the class of symbols $\Sigma^0_{k,l}(\mathbb{R}^n)(\equiv S(1, g^{k,l}))$.

Associated Sobolev spaces

Using the functional calculus on (the compact, positive operator) $\mathcal{A}$ as in (1), we define the operator $\mathcal{A}^{\frac{m}{2}}$, for $m \in \mathbb{R}$, by

$$\mathcal{A}^{\frac{m}{2}} u = \sum_{j=1}^{\infty} \lambda_j^{\frac{m}{2}} (\phi_j, u)_{L^2} \phi_j \,, \quad \text{for} \quad u \in Dom\left(\mathcal{A}^{\frac{m}{2}}\right) ,$$

where $(\phi_j)_{j\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R}^n)$ made of eigenfunctions of $\mathcal{A}$, and $(\lambda_j)_{j\in\mathbb{N}}$ the corresponding eigenvalues.

The **Sobolev spaces related to** $\mathcal{A}$, denoted by $\mathcal{A}^m_{k,l}(\mathbb{R}^n)$, for $m \in \mathbb{R}$, $k, l \geq 1$, integers, is the subspace $\mathcal{S}'(\mathbb{R}^n)$ that is the completion of $Dom\left(\mathcal{A}^{\frac{m}{2}}\right)$ for the norm

$$\|u\|_{\mathcal{A}_m} := \|\mathcal{A}^{\frac{m}{2}} u\|_{L^2(\mathbb{R}^n)} .$$

Continuity Properties

The identification of the Sobolev spaces $A^m_{k,l}(\mathbb{R}^n)$ with suitable Sobolev spaces $H(M, g)$ in the Weyl-Hörmander setting, and the general theory yield:

- Let $m \in \mathbb{R}$ and $k, l \geq 1$ integers. If $a \in \Sigma^m_{k,l}(\mathbb{R}^n)$, then

$$Op^w(a) : \mathcal{A}^s_{k,l}(\mathbb{R}^n) \xrightarrow{\text{cont.}} \mathcal{A}^{s-m}_{k,l}(\mathbb{R}^n) , \quad \forall s \in \mathbb{R} .$$

More generally, equivalence of quantizations in our particular case yield:

- For $a \in \Sigma^m_{k,l}(\mathbb{R}^n)$ $(m \in \mathbb{R}, k, l \geq 1)$ we have

$$Op^\tau(a) : \mathcal{A}^s_{k,l}(\mathbb{R}^n) \xrightarrow{\text{cont.}} \mathcal{A}^{s-m}_{k,l}(\mathbb{R}^n) , \quad \forall s \in \mathbb{R} , \forall \tau \in \mathbb{R} .$$

Anarmonic oscillators and Schatten-classes of operators

For $1 \leq r < \infty$, we denote by $S_r(L^2(\mathbb{R}^n))$ the r^{th}-Shatten class of operators.

- For $g = g^{p,q}$ as in (3), and for $a \in S(\lambda_g^{-\mu}, g)$, $\mu > \frac{n}{r}$, we have

$$Op^\tau(a) \in S_r(L^2(\mathbb{R}^n)) , \quad \text{for all} \quad \tau \in \mathbb{R} .$$

- For the operator $\mathcal{A}$ as in (1), or more generally, for the operator T as in (2), and for $\mu > \frac{n(k+l)}{2klr}$ (for k, l as in (1), or accordingly to the choices of p, q as in (2)) we have

$$T^{-\mu}, \mathcal{A}^{-\mu} \in S_r(L^2(\mathbb{R}^n)) .$$

Eigenvalue asymptotics for negative powers of operators

The general theory on the Schatten-classes $S_r(L^2(\mathbb{R}^n))$ and the Weyl-inequality (see [5])

$$\sum_{j=1}^{\infty} | \underbrace{\lambda_j(T)^r}_{\text{eigen. of } T} | \leq \sum_{j=1}^{\infty} \underbrace{s_j(T)^r}_{\text{sing. val. of } T} , \quad r > 0 ,$$

imply:

- For $g = g^{p,q}$ as in (3), and for $a \in S(\lambda_g^{-\mu}, g)$, $\mu > \frac{n}{r}$, and for any $\tau \in \mathbb{R}$ we have

$$\lambda_j(Op^\tau(a)) = o(j^{-\frac{1}{r}}) , \quad \text{as} \quad j \to \infty .$$

- For the operator $\mathcal{A}$ as in (1), or more generally, for the operator T as in (2), and for $\mu > \frac{n(k+l)}{2klr}$ (for k, l as in (1), or accordingly to the choices of p, q as in (2)) we have

$$\lambda_j(\mathcal{A}^{-\mu}) = o(j^{-r}) , \quad \text{as} \quad j \to \infty .$$

Rate of growth of the eigenvalues of the anharmonic oscillator $\mathcal{A}$

Let k, l be as in (1) and $r > \frac{n(k+l)}{2kl}$. Then for every $N \in \mathbb{N}$ there exists $N_0 \in \mathbb{N}$ such that

$$N j^{\frac{1}{r}} \leq \lambda_j(\mathcal{A}) , \quad \text{for} \quad j \geq N_0 .$$

Thus, the eigenvalues $\lambda_j(\mathcal{A})$ are at least of growth

$$j^{\frac{1}{r}} , \quad \text{as} \quad j \to \infty .$$

Asymptotic expansions of $\lambda_j(\mathcal{A})$ have also been studied in [1] and [2].

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