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Gelfand triples for the Kohn-Nirenberg quantization on a homogeneous Lie group

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Abstract

A Gelfand triple is a triple of spaces (E, H, E') such that H is a Hilbert space and there are continuous dense embeddings $E \hookrightarrow H \hookrightarrow E'$. A classical example is $(\mathcal{S}(G), L^2(G), \mathcal{S}'(G))$ in which $\mathcal{S}(G)$ is the space of rapidly decreasing functions on a nilpotent Lie group G. But characterizing the image of $\mathcal{S}(G)$ under the group Fourier transform $\mathcal{F}_G$ is difficult and unwieldy. For this purpose we define and study new Gelfand triples for the Fourier transform and the Kohn-Nirenberg quantization.
The new Gelfand triples contain distribution spaces for which we may define multiplication on the Fourier side for a large class of smooth operator valued functions.
In this regime we may also show that the formula $a = \rho^* \cdot (A \otimes 1)(\rho)$ for the Kohn-Nirenberg symbol a holds for any operator $A \in \mathcal{L}(\mathcal{O}_{\mathrm{M}}(G))$.

Gelfand triples for the Fourier transform

Homogeneous Lie groups

Let G be a homogeneous Lie group, i.e. G is a nilpotent Lie group together with an diagonalizable derivation A on the Lie algebra $\mathfrak{g}$ with only positive eigenvalues. The group of automorphisms

$$\mathbb{R}^+ \to \mathrm{Hom}(G)\colon \lambda \mapsto \delta_\lambda \quad \text{where} \quad \delta_\lambda x = \mathrm{e}^{A \log \lambda} \log_G(x)$$

are the dilations on G. The trace $Q = \mathrm{tr}(A)$ is the *homogeneous dimension* of G. Let $\widehat{G}$ be the dual of G, i.e. the set of unitary equivalence classes of irreducible unitary representations. Since G is nilpotent we may use Kirillov's correspondence between elements of $\widehat{G}$ and coadjoint orbits.

The group Fourier transform for $\dim Z(G) = 1$ and $\mathrm{SI}/\mathrm{Z} \neq \emptyset$

Let π be a fixed irreducible unitary representation that is square integrable modulo the center (in symbols $\pi \in \mathrm{SI}/\mathrm{Z}$). Furthermore we choose a real structure C_π on $(H_\pi^\infty, H_\pi, H_\pi^{-\infty})$, where H_π is the representation space of π, H_π^∞ is the space of smooth vectors of π and $H_\pi^{-\infty} = (H_\pi^\infty)'$. We will write

$$\pi_\lambda(x) := \pi(\delta_\lambda x),\ \lambda > 0 \quad \text{and} \quad \pi_\lambda(x) := C_\pi\, \pi_{-\lambda}(x)\, C_\pi,\ \lambda \in \mathbb{R}^\times := \mathbb{R} \setminus \{0\}.$$

Using this definition, we may define the following isomorphism of measure spaces

$$(\mathbb{R}^\times, c_\pi |\lambda|^{Q-1} \mathrm{d}\lambda) \to (\widehat{G}_{\mathrm{gen}}, \widehat{\mu})\colon \lambda \mapsto [\pi_\lambda], \tag{1}$$

in which $\widehat{G}_{\mathrm{gen}} \subset \widehat{G}$ are the equivalence classes corresponding to generic orbits and $\widehat{\mu}$ is the Plancherel measure. Since $\widehat{\mu}(\widehat{G} \setminus \widehat{G}_{\mathrm{gen}}) = 0$, we may use (1) and Pedersen's machinery [3] to construct a unitary isomorphism

$$j_\pi\colon L^2(\mathbb{R}^\times, c_\pi |\lambda|^{Q-1} \mathrm{d}\lambda) \mathbin{\widehat{\otimes}_{\mathrm{HS}}} \mathcal{HS}(H_\pi) \to L^2(\widehat{G}) = \mathcal{F}_G L^2(G).$$

We will use j_π to define an adjusted Fourier transform as described in Figure 1.

Defining new Gelfand triples

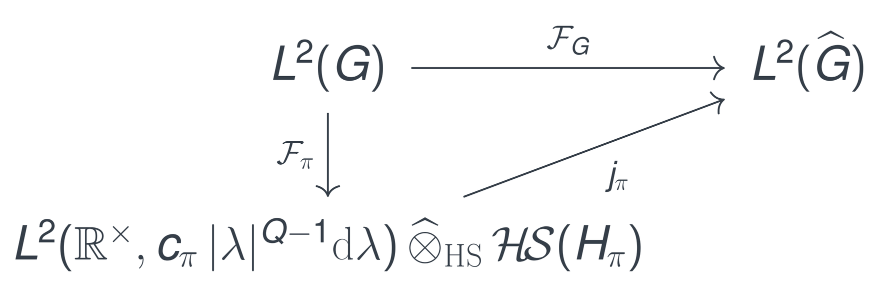

Figure 1: Commutative diagram describing $\mathcal{F}_\pi$

But it is rather inconvenient that the Fourier image of the space of Schwartz functions $\mathcal{S}(G)$ does not decompose into a tensor product of a functions space and an operator space. In order to bypass the problematic behaviour of $\mathcal{F}_\pi \varphi(\lambda)$ in $\lambda \to 0$ for $\varphi \in \mathcal{S}(G)$, we will use operator valued functions that vanish in $\lambda = 0$. Denote by $\mathcal{S}_*(\mathbb{R})$ the space of Schwartz functions on $\mathbb{R}$ that are orthogonal to all polynomials. We will also denote by $\mathcal{S}(\mathbb{R}^\times)$ the set of Schwartz functions on $\mathbb{R}^\times$ that vanish of arbitrarily high order in $\lambda \to 0$. We split the Lie algebra into vector subspaces $\mathfrak{g} = Z(\mathfrak{g}) \oplus \omega$ and define

$$\mathcal{S}_*(G) := \{\varphi \in \mathcal{S}(G) \mid [(\lambda, x) \mapsto \varphi \circ \exp_G(\lambda z + x)] \in \mathcal{S}_*(\mathbb{R}) \mathbin{\hat{\otimes}_\varepsilon} \mathcal{S}(\omega)\}$$

for any $z \in Z(\mathfrak{g}) \setminus \{0\}$. Using this definition we may formulate the following theorem.

Theorem

The map $\mathcal{F}_\pi$ is an isomorphism in each row of

$$\mathcal{F}_\pi\colon \begin{pmatrix} \mathcal{S}_*(G) \\ L^2(G) \\ \mathcal{S}_*'(G) \end{pmatrix} \to \begin{pmatrix} \mathcal{S}(\mathbb{R}^\times) \mathbin{\widehat{\otimes}_\varepsilon} \mathcal{L}(H_\pi^{-\infty}, H_\pi^\infty) \\ L^2(\mathbb{R}^\times, c_\pi |\lambda|^{Q-1} \mathrm{d}\lambda) \mathbin{\widehat{\otimes}_{\mathrm{HS}}} \mathcal{HS}(H_\pi) \\ \mathcal{S}'(\mathbb{R}^\times) \mathbin{\widehat{\otimes}_\varepsilon} \mathcal{L}(H_\pi^\infty, H_\pi^{-\infty}) \end{pmatrix} =: \begin{pmatrix} \mathcal{S}(\mathbb{R}^\times; \pi) \\ L^2(\mathbb{R}^\times; \pi) \\ \mathcal{S}'(\mathbb{R}^\times; \pi) \end{pmatrix}.$$

Multiplication operators on the Fourier side

Due to [1] we may identify a large class of operator valued functions that act as multiplication operators on $\mathcal{S}(\mathbb{R}^\times; \pi)$ and $\mathcal{S}'(\mathbb{R}^\times; \pi)$. In the following $\mathcal{O}_{\mathrm{M}}(\cdot)$ denotes the space of multiplication operators on $\mathcal{S}(\cdot)$.

Theorem

The multiplication $(f, \sigma) \mapsto f \cdot \sigma$, defined by pointwise composition, is a hypocontinuous bilinear map

$$\mathcal{O}_{\mathrm{M}}(\mathbb{R}^\times) \mathbin{\widehat{\otimes}_\varepsilon} \mathcal{L}(H_\pi^\infty) \times \mathcal{S}(\mathbb{R}^\times; \pi) \to \mathcal{S}(\mathbb{R}^\times; \pi)\colon \tag{2}$$

and by transposition there is a hypocontinuous multiplication

$$\mathcal{O}_{\mathrm{M}}(\mathbb{R}^\times) \mathbin{\widehat{\otimes}_\varepsilon} \mathcal{L}(H_\pi^{-\infty}) \times \mathcal{S}'(\mathbb{R}^\times; \pi) \to \mathcal{S}'(\mathbb{R}^\times; \pi). \tag{3}$$

Gelfand triples for the Kohn-Nirenberg quantization

We may see the Kohn-Nirenberg quantization as the map

$$\mathrm{Op}\colon L^2(G) \mathbin{\widehat{\otimes}_{\mathrm{HS}}} L^2(\widehat{G}) \to \mathcal{HS}(L^2(G)), \quad \mathrm{Op}(a) := \mathcal{K}^{-1} \mathcal{T}^{-1} (1 \otimes \mathcal{F}_G^{-1}), \tag{4}$$

in which $\mathcal{T} f(x, y) = f(x, xy^{-1})$ and $\mathcal{K}\colon \mathcal{HS}(L^2(G)) \to L^2(G \times G)$ is the canonical isomorphism with $\langle A\varphi, \psi \rangle = \langle \mathcal{K}(A), \psi \otimes \varphi \rangle$. Let us call Op_π the map we obtain by substituting $\mathcal{F}_G$ by $\mathcal{F}_\pi$. Similar to Figure 1 we get the commutative diagram

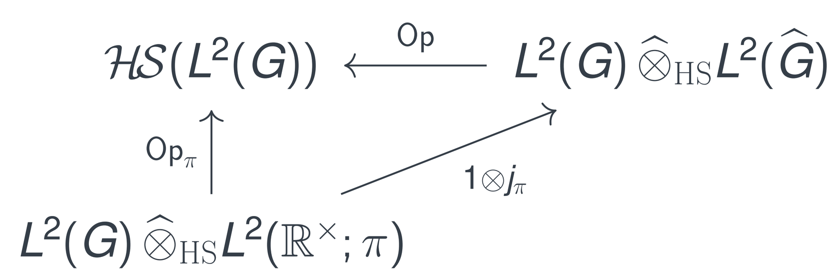

Figure 2: Commutative diagram describing Op_π

As before, we want to construct Gelfand triples such that Op_π becomes an easy to describe isomorphism between Gelfand triples. For this purpose it is enough to realize, that $\mathcal{T}$ restricts to an isomorphism on $\mathcal{S}(G) \mathbin{\widehat{\otimes}_\varepsilon} \mathcal{S}_*(G)$ such that $\langle \mathcal{T} f, g \rangle = \langle f, \mathcal{T}^{-1} g \rangle$. By (4) we may formulate the next theorem.

Theorem

The Kohn-Nirenberg quantization, Op_π, is an isomorphism in each row of

$$\mathrm{Op}_\pi\colon \begin{pmatrix} \mathcal{S}(G) \mathbin{\widehat{\otimes}_\varepsilon} \mathcal{S}(\mathbb{R}^\times; \pi) \\ L^2(G) \mathbin{\widehat{\otimes}_{\mathrm{HS}}} L^2(\mathbb{R}^\times; \pi) \\ \mathcal{S}'(G) \mathbin{\widehat{\otimes}_\varepsilon} \mathcal{S}'(\mathbb{R}^\times; \pi) \end{pmatrix} \to \begin{pmatrix} \mathcal{L}(\mathcal{S}_*'(G), \mathcal{S}(G)) \\ \mathcal{HS}(L^2(G)) \\ \mathcal{L}(\mathcal{S}_*(G), \mathcal{S}'(G)) \end{pmatrix}.$$

On the integral formula for Kohn-Nirenberg operators

We may realize that $\rho\colon (x, \lambda) \mapsto \pi_\lambda(x)$ is an operator valued function in $\mathcal{O}_{\mathrm{M}}(\mathbb{R}^\times \times G) \mathbin{\widehat{\otimes}_\varepsilon} \mathcal{L}(H_\pi^\infty)$. By using the continuity and invertibility of Op_π, the continuous dense embedding $\mathcal{S}_*(G) \hookrightarrow \mathcal{O}_{\mathrm{M}}(G)$ and the hypocontinuous multiplication on $\mathcal{O}_{\mathrm{M}}(G \times \mathbb{R}^\times) \mathbin{\widehat{\otimes}} \mathcal{L}(H_\pi^\infty)$, we are able to prove the following theorem.

Theorem

For any $A \in \mathcal{L}(\mathcal{O}_{\mathrm{M}}(G), E)$, $E \in \{\mathcal{S}(G), \mathcal{O}_{\mathrm{M}}(G)\}$, the equality $a := \rho^* \cdot (A \otimes 1)(\rho) = \mathrm{Op}_\pi^{-1}(A)$ is valid and we may evaluate A by

$$A\varphi = \int_{\mathbb{R}^\times} \mathrm{tr}[\pi_\lambda\, a(\cdot, \lambda)\, \mathcal{F}_\pi \varphi(\lambda)]\, c_\pi |\lambda|^{Q-1} \mathrm{d}\lambda, \quad \textit{for} \quad \varphi \in \mathcal{S}(G),$$

where the integral exists weakly in E. Furthermore

$$\mathrm{Op}_\pi^{-1}\colon \mathcal{L}(\mathcal{O}_{\mathrm{M}}(G)) \to \mathcal{O}_{\mathrm{M}}(G \times \mathbb{R}^\times) \mathbin{\widehat{\otimes}_\varepsilon} \mathcal{L}(H_\pi^\infty)$$

is continuous.

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