Defining new Gelfand triples

Homogeneous Lie groups

Let $G$ be a homogeneous Lie group, i.e., $G$ is a nilpotent Lie group together with an diagonalizable derivation $A$ on the Lie algebra $g$ with only positive eigenvalues. The group of automorphisms

$$\mathbb{R}^+ \to \text{Hom}(G): \lambda \mapsto \lambda \cdot G \text{, where } \delta_t(x) = e^{A_1 \log(x)}$$

are the dilations on $G$. The trace $Q = \text{tr}(A)$ is the homogeneous dimension of $G$. Let $G$ be the dual of $G$, i.e., the set of unitary equivalence classes of irreducible unitary representations. Since $G$ is nilpotent we may use Kirillov’s correspondence between elements of $G$ and coadjoint orbits.

The group Fourier transform for $\dim Z(G) = 1$ and $\mathbb{Z}/2 \neq 0$

Let $\pi$ be a fixed irreducible unitary representation that is square integrable modulo the center (in symbols $\pi \in \mathbb{Z}/2$). Furthermore we choose a real structure $C_1$ on $(H^\pi, H^\ast_\pi)$, where $H^\pi$ is the representation space of $\pi$, $H^\ast_\pi$ is the space of smooth vectors of $\pi$ and $H^\ast_\pi = (H^\pi)\ast$. We will write $\pi(x) := C_1(x)$, $x \in \mathbb{R}^+$ and $\pi_0(x) := C_1(x) \pi_0(x) \in \mathbb{R}^+ \setminus \{0\}$.

Using this definition, we may define the following isomorphism of measure spaces

$$\mathbb{R}^+ \to \text{Hom}(\pi_0): \lambda \mapsto \lambda \cdot \pi_0 \text{, in which } \hat{G}_\lambda \subset G \text{ are the equivalence classes corresponding to open orbits and } \lambda \text{ is the Plancherel measure. Since } \mu_0(G, \hat{G}_\lambda) = 0, \text{ we may use (1) and Pedersen’s machinery [3] to construct a unitary isomorphism }$$

$$f : L^2(\mathbb{R}^+, \mathbb{C}, |\cdot|^{-1} \ast_1) \overset{\sim}{\to} L^2(\lambda \cdot \pi_0).$$

We will use $f$ to define an adjusted Fourier transform as described in Figure 1.

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$$L^2(G) \xrightarrow{F_\lambda} L^1(\hat{G})$$

But it is rather inconvenient that the Fourier image of the space of Schwartz functions $S(G)$ does not decompose into a tensor product of a functions space and an operator space. In order to bypass the problematic behaviour of $F_\lambda$ in $\lambda \to 0$ for $\varphi \in S(G)$, we will use operator valued functions that vanish in $\lambda \to 0$. Denote by $S_r(S)$ the space of Schwartz functions on $S$ that are orthogonal to all polynomials. We will also denote by $S_r(\mathbb{R}^n)$ the set of Schwartz functions on $\mathbb{R}^n$ that vanish of arbitrarily high order in $\lambda \to 0$.

We split the Lie algebra into vector subspaces $g = Z(g) \oplus \omega$ and define

$$S_r(\xi) := \{ \varphi \in S(g) \mid \varphi \text{ and } e^{A_0} (x + i) \varphi \in S_r(\mathbb{R}^n) \} \text{ for } x \in Z(G) \cap \{0\}.$$