PROPAGATION OF SINGULARITIES ON HYPO-ANALYTIC STRUCTURES

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GEVREY VECTORS

Let Ω be a smooth manifold of dimension $m > 0$ and consider $\{\mathrm{L}_1, \dots, \mathrm{L}_n\}$, pairwise commuting complex vector fields defined on Ω, where $n \leq m$. Let $s \geq 1$, we say that a distribution $u \in \mathcal{D}'(\Omega)$ is a **Gevrey-s vector** for $\{\mathrm{L}_1, \dots, \mathrm{L}_n\}$ if $u \in \mathcal{C}^\infty(\Omega)$ and for every $K \subset \Omega$, compact set, there is a positive constant $C = C_K$ such that

$$\sup_{x \in K} |\mathrm{L}^\alpha u(z)| \leq C^{|\alpha|+1} \alpha!^s, \quad \forall \alpha \in \mathbb{Z}_+^m,$$

and we write $u \in \mathrm{G}^s(\Omega; \mathrm{L}_1, \dots, \mathrm{L}_n)$. If $s = 1$ we write $u \in \mathcal{C}^\omega(\Omega; \mathrm{L}_1, \dots, \mathrm{L}_n)$, and we say that u is an **analytic vector** for $\{\mathrm{L}_1, \dots, \mathrm{L}_n\}$.

HYPO-ANALYTIC STRUCTURES

Let Ω be a smooth manifold of dimension $n+m$. A hypo-analytic structure on Ω of corank m is a family $\{U_\alpha, (Z_{\alpha,1}, \cdots, Z_{\alpha,m})\}_{\alpha \in \Lambda}$ satisfying the following conditions:

- $\{U_\alpha\}_{\alpha \in \Lambda}$ is an open covering of Ω;
- $Z_{\alpha,j} : U_\alpha \longrightarrow \mathbb{C}$ is a $\mathcal{C}^\infty$ function for every $j = 1, \cdots, m$ and $\alpha \in \Lambda$;
- $\mathrm{d}Z_{\alpha,1}, \cdots, \mathrm{d}Z_{\alpha,m}$ are $\mathbb{C}$-linearly independent at each point of U_α, for every $\alpha \in \Lambda$
- If $U_\alpha \cap U_\beta \neq \emptyset$ then for every $p \in U_\alpha \cap U_\beta$ there exists a biholomorphic map $F^p_{\alpha,\beta}$ such that $Z_\beta = F^p_{\alpha,\beta} \circ Z_\alpha$ in some neighborhood of p in $U_\alpha \cap U_\beta$.

We call a pair (U, Z) a hypo-analytic chart. We can associate it a locally integrable structure $\mathcal{V}$ setting its orthogonal, T', as the complex bundle defined, locally, by the differentials $\mathrm{d}Z_1, \dots, \mathrm{d}Z_m$.

EXAMPLE

Let $M \subset \mathbb{C}^N$ be a **generic CR** submanifold of codimension d, so the CR dimension is equal to $N - d$. We can associate a hypo-analytic structure on M setting $Z = z|_M$, for every holomorphic coordinate system (U, z). In this case the locally integrable structure $\mathcal{V}$ is equal to $\mathrm{T}^{0,1}M$, *i.e.*, the collection of all anti-holomorphic vector fields that are tangent to M.

THE LOCAL SET UP AND THE REAL STRUCTURE BUNDLE

For simplicity assume that $0 \in \Omega$. There is a hypo-analytic chart $(U, Z_1(x,t), \dots, Z_m(x,t))$ centered at 0 with $U = V \times W$, where $V \subset \mathbb{R}^m$ and $W \subset \mathbb{R}^n$ are open balls centered at the origin, and the map $Z = (Z_1, \dots, z_m)$ is given by

$$Z_j(x,t) = x_j + i\phi_j(x,t), \qquad j = 1, \dots, m,$$

with $(x,t) \in V \times W$, where the map $\phi(x,t) = (\phi_1(x,t), \dots, \phi_m(x,t))$ is smooth, real-valued, $\phi(0) = 0$ and $\mathrm{d}_x\phi(0) = 0$. We associate the complex vector fields $\{\mathrm{M}_1, \dots, \mathrm{M}_m, \mathrm{L}_1, \dots, \mathrm{L}_n\}$ with the following properties:

$$\begin{array}{ll} \mathrm{L}_j Z_k = 0 & \mathrm{M}_j Z_k = \delta_{j,k} \\ \mathrm{L}_j t_k = \delta_{j,k} & \mathrm{M}_j t_k = 0. \end{array}$$

For every $t \in W$ the set $\Sigma_t = \{Z(x,t) \,:\, x \in V\}$ is a maximally real submanifold of $\mathbb{C}^m$. The **real structure bundle** of Σ_t, $\mathbb{R}\mathrm{T}'_{\Sigma_t}$, is given by

$$\mathbb{R}\mathrm{T}'_{\Sigma_t} = \{(Z(x,t), {}^{\mathrm{t}}Z_x(x,t)^{-1}\xi) \,:\, x \in V, \xi \in \mathbb{R}^m\}.$$

AN FBI CHARACTERIZATION OF GEVREY VECTORS

Let $\mathcal{H} \subset \mathbb{C}^m$ be a maximally real submanifold and $u \in \mathcal{E}'(\mathcal{H})$. We define the FBI transform of u by

$$\mathfrak{F}[u](z,\zeta) \doteq \left\langle u(z'), e^{i\zeta\cdot(z-z') - \langle\zeta\rangle(z-z')^2} \Delta(z - z', \zeta) \right\rangle,$$

for $z \in \mathbb{C}^m$ and $\zeta \in \mathfrak{C}_1 = \{\eta \in \mathbb{C}^m \,:\, |\mathrm{Im}\,\eta| < |\mathrm{Re}\,\eta|\}$, where $\Delta(z,\zeta) = \det(\mathrm{Id} + i(z \odot \zeta)/\langle\zeta\rangle)$. Now if u is a distribution on U (the same as before) such that $\mathrm{L}_j u \in \mathrm{G}^s(U; \mathrm{L}_1, \dots, \mathrm{L}_n, \mathrm{M}_1, \dots, \mathrm{M}_m)$, then actually $u \in \mathcal{C}^\infty(W; \mathcal{D}'(W))$. So we define the FBI transform of $u \in \mathcal{C}^\infty(W; \mathcal{E}'(V))$ by

$$\mathfrak{F}[u](t; z, \zeta) = \mathfrak{F}[u_t^*](z,\zeta),$$

where u_t^* is the compactly supported distribution on Σ_t defined by $u_t^*(z) = u(x,t)$, for $z = Z(x,t)$.

Theorem 1 (N. Braun Rodrigues, 2020) *Let $u \in \mathcal{C}^\infty(W; \mathcal{D}'(V))$ be a solution of*

$$\begin{cases} \mathrm{L}_1 u = f_1, \\ \quad\vdots \\ \mathrm{L}_n u = f_n, \end{cases}$$

where $f_j \in \mathrm{G}^s(U; \mathrm{L}_1, \cdots, \mathrm{L}_n, \mathrm{M}_1, \dots, \mathrm{M}_m)$, $j = 1, \dots, n$. Then are equivalent:

1. *$u|_{U_0} \in \mathrm{G}^s(U; \mathrm{L}_1, \cdots, \mathrm{L}_n, \mathrm{M}_1, \dots, \mathrm{M}_m)$, for some open neighborhood of the origin U_0;*
2. *For every $\chi \in \mathcal{C}_c^\infty(V)$, with $0 \leq \chi \leq 1$ and $\chi \equiv 1$ in some open neighborhood of the origin, there exist $\widetilde{V} \subset V$, $\widetilde{W} \subset W$, open balls centered at the origin, constants $C, \epsilon > 0$ such that*

$$|\mathfrak{F}[\chi u](t; z, \zeta)| \leq C e^{-\epsilon|\zeta|^{\frac{1}{s}}}, \qquad \forall t \in \widetilde{W}, (z,\zeta) \in \mathbb{R}\mathrm{T}'_{\Sigma_t}\big|_{\widetilde{V}} \setminus 0,$$

where $(z,\zeta) \in \mathbb{R}\mathrm{T}'_{\Sigma_t}\big|_{\widetilde{V}}$ means that $z = Z(x,t)$, $\zeta = {}^{\mathrm{t}}Z_x(x,t)^{-1}\xi$, $\xi \in \mathbb{R}^m \setminus 0$ and $x \in \widetilde{V}$.

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"PROPAGATORS"

We shall consider only analytic tube structures, *i.e.*, locally the hypo-analytic structure is given by $Z(x,t) = x + i\phi(t)$, defined on $U = V \times W$, and $\phi(t)$ is analytic. One of the reasons we are only dealing with tube structures is that the real structure bundle is trivial, *i.e.*,

$$\mathbb{R}\mathrm{T}'_{\Sigma_t} = \{(Z(x,t), \xi) \,:\, x \in V, \xi \in \mathbb{R}^m\},$$

for every $t \in W$. Let $\Sigma \subset \Omega$ be a connected submanifold of Ω, satisfying the following properties:

1. For every $p \in \Sigma$ there is (U, Z), a hypo-analytic chart, with $p \in U$, such that $\Sigma \cap U \subset Z^{-1}(0)$;
2. In the same situation as above, for every $q \in \Sigma \cap U$, and $\widetilde{U}_1 \Subset U$, an open neighborhood of p, there is $\widetilde{U}_2 \Subset U$, an open neighborhood of q, such that the connected component of the fiber $Z^{-1}(Z(q'))$ that contains q' intersects $\widetilde{U}_1$, for every $q' \in \widetilde{U}_2$;
3. the map $\Sigma \ni p \mapsto \sup\{r > 0 \,:\, B_r(p) \subset U\}$ is continuous.

PROPAGATION OF SINGULARITIES

Let $u \in \mathcal{D}'(\Omega)$. We say that $\mathbb{L}u \in \mathrm{G}^s$ if for every (U, Z) described in the local setup section we have that

$$\mathrm{L}_j u \in \mathrm{G}^s(U; \mathrm{L}_1, \cdots, \mathrm{L}_n, \mathrm{M}_1, \dots, \mathrm{M}_m),$$

for $j = 1, \dots, n$. Note that since the structure is real-analytic, this is equivalent to $\mathrm{L}_j u \in \mathrm{G}^s(U)$.

Theorem 2 (N. Braun Rodrigues, 2020)
In the same situation as above, if $u \in \mathcal{D}'(\Omega)$ is such that $\mathbb{L}u \in \mathrm{G}^s(\Omega)$, then $\mathrm{singsupp}_s\, u \cap \Sigma = \emptyset$ or $\Sigma \subset \mathrm{singsupp}_s\, u$.

REFERENCES

[1] Nicholas Braun Rodrigues. An FBI characterization for Gevrey vectors on hypo-analytic structures and propagation of Gevrey singularities. *arXiv:2006.04590*, 2020.