

Abstract

In what follows we shall present some recent results about the validity of smoothing and Strichartz-type estimates for time-degenerate Schrödinger operators. These results have important applications in the study of the local well-posedness of the initial value problem (IVP) associated with the operators under consideration.

Time degenerate Schrödinger-type Operators

We shall consider the following classes of degenerate Schrödinger-type operators

$$\mathcal{L}_\alpha = i\partial_t + t^\alpha \Delta_x + b(t, x) \cdot \nabla_x, \quad (1)$$

$$\mathcal{L}_c = i\partial_t + c'(t) \Delta_x, \quad (2)$$

where $\alpha > 0$, $b(t, x) = (b_1(t, x), \dots, b_n(t, x))$ is such that, for all $j = 1, \dots, n$, $b_j \in C([0, T], C_b^\infty(\mathbb{R}^n))$, while $c \in C^1(\mathbb{R})$.

The class (1) was considered in [1] where both homogeneous and inhomogeneous weighted local smoothing estimates are derived. These estimates are also employed to obtain local well-posedness results for the associated nonlinear IVP (see [1]).

The class (2) was studied in [2] where global weighted homogeneous smoothing estimates are proved by means of comparison principles. For the class (2) weighted Strichartz estimates are proved in [2] where the application to the local well-posedness of the semilinear IVP is given.

The main difference between the operators of the form (1) and (2) and the other Schrödinger operators studied so far is the **presence of degeneracies**. Specifically, the degeneracies are given by the coefficient t^α and $c'(t)$ in (1) and (2) respectively.

Why to study smoothing and Strichartz estimates?

These estimates give us important information about the regularity properties of the solution of the IVP. In particular:

- **The homogeneous smoothing** effect describes a gain of smoothness of the homogeneous solution of the IVP with respect to the smoothness of the initial data.
- **The inhomogeneous smoothing** effect describes a gain of smoothness of the solution of the inhomogeneous IVP with respect to the regularity of the inhomogeneous data.
- **Strichartz estimates** describe a gain of integrability instead of a gain of smoothness of the solution of the IVP.

Additionally, **these estimates are fundamental in order to prove the local well-posedness of the corresponding semilinear and nonlinear IVP** through the standard fixed point argument.

What are comparison principles? (Following [3])

Question: Given two operators $P_a(t, x, D_t, D_x)$ and $P_b(t, x, D_t, D_x)$ depending on two functions a and b respectively, is it possible to compare (in a suitable sense) the solutions of the HIVP (homogeneous IVP) for P_a and P_b if a and b are comparable (in a suitable sense)?

This is essentially what the **comparison principles** we refer to do, that is, **they translate a relation between a and b in a relation between the solutions of the HIVP for P_a and P_b** .

Example (see [3]). Let $a, \tilde{a} \in C^1(\mathbb{R})$ be real valued and strictly monotone on the support of a measurable function χ , and let $\sigma, \tau \in C^0(\mathbb{R})$. Then, if $\forall \xi \in \text{supp } \chi$ we have

$$\frac{|\sigma(\xi)|}{|\alpha'(\xi)|^{1/2}} \leq C \frac{|\tau(\xi)|}{|\tilde{\alpha}'(\xi)|^{1/2}}, \quad \text{then } \|\chi(D_x)\sigma(D_x)e^{i\tilde{\alpha}(D)}\varphi\|_{L^2(\mathbb{R}^2)} \leq C\|\chi(D_x)\tau(D_x)e^{i\alpha(D)}\varphi\|_{L^2(\mathbb{R}^2)}$$

for all $\varphi = \varphi(x)$ smooth enough.

Weighted local smoothing effect for \mathcal{L}_α

We consider the IVP

$$\begin{cases} \partial_t u = i t^\alpha \Delta_x u + i b(t, x) \cdot \nabla_x u + f(t, x) \\ u(0, x) = u_0(x). \end{cases} \quad (3)$$

When $b \equiv 0$ one can proceed by using Fourier analysis. However, in the general case $b \neq 0$, the use of pseudo-differential calculus is needed.

Theorem (F.-Staffilani)

Let $u_0 \in H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$. Assume that, for all $j = 1, \dots, n$, b_j is such that $b_j \in C([0, T], C_b^\infty(\mathbb{R}^n))$ and there exists $\sigma > 1$ such that

$$|\text{Im } \partial_x^j b_j(t, x)|, |\text{Re } \partial_x^j b_j(t, x)| \lesssim t^\alpha \langle x \rangle^{-\sigma-|\gamma|}, \quad x \in \mathbb{R}^n, \quad (4)$$

and denote by $\lambda(|x|) := \langle x \rangle^{-\sigma}$ and by $\Lambda := \langle \xi \rangle$.

Then

- (i) If $f \in L^1([0, T]; H^s(\mathbb{R}^n))$ then the IVP (3) has a unique solution $u \in C([0, T]; H^s(\mathbb{R}^n))$ and there exist positive constants C_1, C_2 such that

$$\sup_{0 \leq t \leq T} \|u(t)\|_s \leq C_1 e^{C_2 \frac{T^{\alpha+1}}{\alpha+1}} \left(\|u_0\|_s + \int_0^T \|f(t)\|_s dt \right);$$

- (ii) If $f \in L^2([0, T]; H^s(\mathbb{R}^n))$ then the IVP (3) has a unique solution $u \in C([0, T]; H^s(\mathbb{R}^n))$ and there exist two positive constants C_1, C_2 such that

$$\begin{aligned} \sup_{0 \leq t \leq T} \|u(t)\|_s^2 &\leq C_1 \int_0^T \int_{\mathbb{R}^n} t^\alpha |\Lambda^{s+1/2} u|^2 \lambda(|x|) dx dt \\ &\leq C_1 e^{C_2 \frac{T^{\alpha+1}}{\alpha+1}} \left(\|u_0\|_s^2 + \int_0^T \|f(t)\|_s^2 dt \right); \end{aligned}$$

- (iii) If $\Lambda^{s-1/2} f \in L^2([0, T] \times \mathbb{R}^n; t^{-\alpha} \lambda(|x|)^{-1} dt dx)$ then the IVP (3) has a unique solution $u \in C([0, T]; H^s(\mathbb{R}^n))$ and there exist positive constants C_1, C_2 such that

$$\begin{aligned} \sup_{0 \leq t \leq T} \|u(t)\|_s^2 &+ \int_0^T \int_{\mathbb{R}^n} t^\alpha |\Lambda^{s+1/2} u|^2 \lambda(|x|) dx dt \\ &\leq C_1 e^{C_2 \frac{T^{\alpha+1}}{\alpha+1}} \left(\|u_0\|_s^2 + \int_0^T \int_{\mathbb{R}^n} t^{-\alpha} \lambda(|x|)^{-1} |\Lambda^{s-1/2} f|^2 dx dt \right). \end{aligned}$$

Local well-posedness results for \mathcal{L}_α

We consider the nonlinear IVPs

$$\text{IVP1} = \begin{cases} \mathcal{L}_\alpha u = \pm u|u|^{2k} \\ u(0, x) = u_0(x), \end{cases} \quad \text{IVP2} = \begin{cases} \mathcal{L}_\alpha u = \pm t^\beta \nabla_x \cdot u^{2k}, \beta \geq \alpha > 0, \\ u(0, x) = u_0(x). \end{cases}$$

Theorem (F.-Staffilani). Let \mathcal{L}_α be such that condition (4) is satisfied. Then the IVP1 is locally well posed in H^s for $s > n/2$.

Theorem (F.-Staffilani). Let \mathcal{L}_α be such that condition (4) is satisfied with $\sigma = 2N$ (thus $\lambda(|x|) = \langle x \rangle^{-2N}$) for some $N \geq 1$, and $s > n + 4N + 3$ such that $s - 1/2 \in 2\mathbb{N}$. Let $H_x^s := \{u_0 \in H^s(\mathbb{R}^n); \lambda(|x|)u_0 \in H^s(\mathbb{R}^n)\}$, then the IVP2 with $\beta \geq \alpha > 0$, is locally well posed in H_x^s .

Global weighted smoothing and Strichartz estimates for \mathcal{L}_c

For operators of the form \mathcal{L}_c several comparison principles are proved in [2]. These are used to obtain global smoothing estimates. We state below only one of them, namely the one corresponding to the suitable generalization of the standard (corresponding to the case $c'(t) = 1$) global homogeneous smoothing estimate. For more global smoothing estimates see [2].

Theorem (F.-Ruzhansky)

Let $n \geq 1$, $c \in C^1(\mathbb{R})$ be such that it vanishes at 0. Then, $\forall x \in \mathbb{R}^n$,

- (i) If c is such that $\{t \in \mathbb{R}; c'(t) = 0\}$ is finite, then there exist a constant $C > 0$ such that, for all j ,

$$\sup_x \| |c'(t)|^{1/2} |D_{x_j}|^{1/2} e^{ic(t)\Delta_x} \varphi \|_{L^2(\mathbb{R}_t^1 \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L_x^2(\mathbb{R}^n)}, \quad \forall \varphi \in L_x^2(\mathbb{R}^n);$$

- (ii) If c is such that the set $\{t \in \mathbb{R}; c'(t) = 0\}$ is countable, then there exists a function $\tilde{c} \in C(\mathbb{R})$, and a positive constant C such that, for all j ,

$$\sup_x \| |\tilde{c}(t)c'(t)|^{1/2} |D_{x_j}|^{1/2} e^{ic(t)\Delta_x} \varphi \|_{L^2(\mathbb{R}_t^1 \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L_x^2(\mathbb{R}^n)}, \quad \forall \varphi \in L_x^2(\mathbb{R}^n);$$

where $x' = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$.

Both global and local Strichartz estimates are satisfied by \mathcal{L}_c . Below we give the statement of the local ones which are those employed to prove the local well-posedness of the semilinear IVP.

Theorem (F.-Ruzhansky)

Let $c \in C^1([0, T])$ be vanishing at 0 and such that $\#\{t \in [0, T]; c'(t) = 0\} = k \geq 1$. Then, on denoting by $L_t^q L_x^p := L^q([0, T]; L^p(\mathbb{R}^n))$, we have that for any (q, p) admissible pair $(\frac{2}{q} + \frac{n}{p} = \frac{n}{2})$ such that $2 < q, p < \infty$, the following estimates hold

$$\| |c'(t)|^{1/q} e^{ic(t)\Delta_x} \varphi \|_{L_t^q L_x^p} \leq C(n, q, p, k) \|\varphi\|_{L_x^2(\mathbb{R}^n)},$$

$$\| e^{ic(t)\Delta_x} \varphi \|_{L_t^q L_x^2} \leq \|\varphi\|_{L_x^2(\mathbb{R}^n)},$$

$$\| |c'(t)|^{1/q} \int_0^t |c'(s)| e^{i(c(t)-c(s))\Delta_x} g(s) ds \|_{L_t^q L_x^p} \leq C(n, q, p, k) \| |c'|^{1/q} g \|_{L_t^q L_x^p},$$

$$\| \int_0^t |c'(s)| e^{i(c(t)-c(s))\Delta_x} g(s) ds \|_{L_t^q L_x^2} \leq C(n, q, p, k) \| |c'|^{1/q} g \|_{L_t^q L_x^p}.$$

Local well-posedness of the semilinear IVP for \mathcal{L}_c

We can now apply the previous results to obtain the local well-posedness of the semilinear IVP

$$\begin{cases} \partial_t u + ic'(t)\Delta_x u = \mu |c'(t)| |u|^{p-1} u, \quad \mu \in \mathbb{R}, \\ u(0, x) = u_0(x). \end{cases} \quad (5)$$

Theorem (F.-Ruzhansky). Let $1 < p < \frac{4}{n} + 1$ and $c \in C^1([0, \infty))$ be vanishing at 0 and it is either strictly monotone or such that $\#\{t \in [0, T]; c'(t) = 0\}$ is finite for any $T < \infty$. Then for all $u_0 \in L^2(\mathbb{R}^n)$ there exists $T = T(\|u_0\|_2, n, \mu, p) > 0$ such that there exists a unique solution u of the IVP (5) in the time interval $[0, T]$ with $u \in C([0, T]; L_x^q(\mathbb{R}^n)) \cap L_t^q L_x^p([0, T]; L_x^{p+1}(\mathbb{R}^n))$ and $q = \frac{4(p+1)}{n(p-1)}$. Moreover the map $u_0 \mapsto u(\cdot, \cdot)$, locally defined from $L^2(\mathbb{R}^n)$ to $C([0, T]; L^2(\mathbb{R}^n))$, is continuous.

References

- [1] S. Federico and G. Staffilani, Smoothing effect for time-degenerate Schrödinger operators, Preprint ArXiv.
- [2] S. Federico and M. Ruzhansky, Smoothing and Strichartz estimates for degenerate Schrödinger-type equations, Preprint ArXiv.
- [3] M. Ruzhansky and M. Sugimoto, Smoothing properties of evolution equations via canonical transforms and comparison principles, Proc. London Math. Soc. (3) 105 (2012), 393–423.