Explicit representations of solutions for linear fractional differential equations with variable coefficients

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Abstract

Explicit solutions of differential equations of complex fractional orders with continuous variable coefficients are established in [3]. The representations of solutions are given in terms of some convergent infinite series of fractional integro-differential operators, which can be widely and efficiently used for analytic and computational purposes.
In the case of constant coefficients, the solution is given by the multivariate Mittag-Leffler functions and the obtained result extends the Luchko-Gorenflo representation formula [2, Theorem 4.1] to a general class of linear fractional differential equations with variable coefficients, to complex fractional derivatives, and to fractional derivatives with respect to a given function.

Description

One of the many open problems in fractional calculus is to present explicit solutions of fractional differential equations (FDEs) with variable coefficients. Existence and uniqueness results for these type problems can be found in the literature. However, in general, explicit representations of solutions have been an essential gap in such problems. A few papers have been published in this direction in the last fifty years.
A modified method of successive approximations is used to establish a unique analytic solution of general FDEs of complex fractional orders with continuous variable coefficients. We follow some ideas from [4] and [1]. A particular case of the obtained result gives the Luchko-Gorenflo representation formula [2, Theorem 4.1].

Modified fractional derivative with respect to another function ${}^{C}D^{\alpha,\phi}_{0+}f(t)$

Let $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$. Let $D^{\alpha,\phi}_{a+}f(t)$ be the left-sided Riemann-Liouville fractional derivative of a function f with respect to another function ϕ ([4]). Then

$$ {}^{C}D^{\alpha,\phi}_{0+}f(t) = D^{\alpha,\phi}_{0+}\left(f(t) - \sum_{j=0}^{n-1} \frac{f^{[j]}_{\phi}(0)}{j!}(\phi(t)-\phi(0))^{j} \right) $$

where $n = -[-\mathrm{Re}(\alpha)]$ for $\alpha \notin \mathbb{N}$, $n = \alpha$ for $\alpha \in \mathbb{N}$ and

$$ f^{[j]}_{\phi}(t) = \left(\frac{1}{\phi'(t)}\frac{d}{dt}\right)^{j} f(t). $$

Fractional differential equations

We consider the following fractional differential equation with continuous variable coefficients (FDEC):

$$ {}^{C}D^{\beta_0,\phi}_{0+}x(t) + \sum_{i=1}^{m} d_i(t)\,{}^{C}D^{\beta_i,\phi}_{0+}x(t) = h(t), \ \ t \in [0,T], $$

under the initial conditions

$$ \left(\frac{1}{\phi'(t)}\frac{d}{dt}\right)^{k} x(t)\Big|_{t=+0} = 0, \quad k = 0,1,\ldots,n_0-1, $$

where $\beta_i \in \mathbb{C}$, $\mathrm{Re}(\beta_i) > 0$, $i = 0,1,\ldots,m-1$, $\mathrm{Re}(\beta_0) > \mathrm{Re}(\beta_1) > \ldots > \mathrm{Re}(\beta_m) \geqslant 0$ (If $\mathrm{Re}(\beta_m) = 0$, then we assume $\mathrm{Im}(\beta_m) = 0$ as well) and n_i are non-negative integers satisfying $n_i - 1 < \mathrm{Re}(\beta_i) \leqslant n_i$, $n_i = \lfloor \mathrm{Re}\beta_i \rfloor + 1$ (or $n_i = -\lfloor -\mathrm{Re}(\beta_i)\rfloor$), $i = 0,1,\ldots,m$.

Theorem

Let $h, d_i \in C[0,T]$, $i = 1,\ldots,m$. Then the initial value problem FDEC has a unique solution $x \in C^{n_0-1,\beta_0}[0,T]$ and it is given by the formula

$$ x(t) = \sum_{k=0}^{+\infty} (-1)^k I^{\beta_0,\phi}_{0+}\left(\sum_{i=1}^{m} d_i(t) I^{\beta_0-\beta_i,\phi}_{0+}\right)^{k} h(t), $$

whenever $\sum_{i=1}^{m} \|d_i\|_{\max} I^{\beta_0-\beta_i,\phi}_{0+} e^{\nu t} \leqslant C e^{\nu t}$ for some $\nu > 0$, where the constant $0 < C < 1$ does not depend on t. In the case of constant coefficients $d_i(t) = \lambda_i \in \mathbb{C}$ $(i = 1,\ldots,m)$ we have:

$$ x(t) = \int_0^t \phi'(s)(\phi(t)-\phi(s))^{\beta_0-1} \times $$
$$ \times E_{(\beta_0-\beta_1,\ldots,\beta_0-\beta_m),\beta_0}(-\lambda_1(\phi(t)-\phi(s))^{\beta_0-\beta_1}, \cdots, -\lambda_m(\phi(t)-\phi(s))^{\beta_0-\beta_m}) h(s)ds, $$

where $E_{(a_1,\ldots,a_n),b}(z_1,\ldots,z_n)$ is the multivariate Mittag-Leffler function [2, Page 12].

Remark

The solution of the considered fractional differential equation under the following more general initial conditions

$$ \left(\frac{1}{\phi'(t)}\frac{d}{dt}\right)^{k} x(t)\Big|_{t=+0} = c_k \in \mathbb{R}, \ \ k = 0,1,\ldots,n_0-1, $$

follows by the latter theorem and the superposition principle. Several results can be established by the consideration of different possible cases of the complex orders.

References

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