Hyperbolic Cauchy problems with multiplicities

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Overview

1. Motivation and Introduction
2. A bit of history
3. The (t, x)-dependent case: Sobolev well-posedness and representation formulas

Motivation and Introduction

Hyperbolic Cauchy problems with multiplicities (weakly hyperbolic)

$$\begin{cases} D_t^m u + \sum_{j=0}^{m-1} A_{m-j}(t,x,D_x) D_t^j u = 0, & (t,x) \in [0,T] \times \mathbb{R}^n, \\ D_t^{l-1} u(0,x) = g_l(x), & l = 1, \dots, m, \end{cases}$$

where each $A_{m-j}(t,x,D_x)$ is a differential operator of order $m-j$ with continuous (or more regular) coefficients. $(D^\alpha = (-i)^{|\alpha|}\partial^\alpha)$

$$\tau^m + \sum_{j=0}^{m-1} A_{(m-j)}(t,x,\xi)\tau^j = \tau^m + \sum_{j=0}^{m-1} \sum_{|\gamma|=m-j} a_{m-j,\gamma}(t,x)\xi^\gamma \tau^j$$

$$= \prod_{l=1}^{m} (\tau - \tau_l(t,x,\xi)),$$

with $\tau_j(t,x,\xi)$ **real**. The roots might coincide (**multiplicities**)!

Let $\langle D_x \rangle$ be the pseudo-differential operator with symbol $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$. We use the transformation

$$u_k = D_t^{k-1} \langle D_x \rangle^{m-k} u.$$

This makes the Cauchy problem equivalent to the system

$$D_t \begin{pmatrix} u_1 \\ \cdot \\ \cdot \\ u_m \end{pmatrix} = \begin{pmatrix} 0 & \langle D_x \rangle & 0 & \dots & 0 \\ 0 & 0 & \langle D_x \rangle & \dots & 0 \\ \dots & \dots & \dots & \dots & \langle D_x \rangle \\ b_1 & b_2 & \dots & \dots & b_m \end{pmatrix} \begin{pmatrix} u_1 \\ \cdot \\ \cdot \\ u_m \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \cdot \\ f \end{pmatrix},$$

where

$$b_j = A_{m-j+1}(t, x, D_x) \langle D_x \rangle^{j-m},$$

with initial condition

$$u_k|_{t=0} = \langle D_x \rangle^{m-k} g_k, \qquad k = 1, \dots, m.$$

We write the matrix as $A(t,x,D_x)+B(t,x,D_x)$ with symbols

$$A(t,x,\xi)=\left(\begin{array}{ccccc} 0 & \langle\xi\rangle & 0 & \dots & 0\\ 0 & 0 & \langle\xi\rangle & \dots & 0\\ \dots & \dots & \dots & \dots & \langle\xi\rangle\\ b_{(1)}(t,x,\xi) & b_{(2)}(t,x,\xi) & \dots & \dots & b_{(m)}(t,x,\xi)\end{array}\right),$$

$$b_{(j)}(t,x,\xi)=A_{(m-j+1)}(t,x,\xi)\langle\xi\rangle^{j-m},$$

$$B(t,x,\xi)=\left(\begin{array}{ccccc} 0 & 0 & 0 & \dots & 0\\ 0 & 0 & 0 & \dots & 0\\ \dots & \dots & \dots & \dots & 0\\ (b_1-b_{(1)})(t,x,\xi) & \dots & \dots & \dots & (b_m-b_{(m)})(t,x,\xi)\end{array}\right)$$

$$(b_j-b_{(j)})(t,x,\xi)=(A_{m-j+1}-A_{(m-j+1)})(t,x,\xi)\langle\xi\rangle^{j-m}.$$

The eigenvalues of the symbol matrix $A(t,x,\xi)$ are the roots $\tau_j(t,x,\xi)$.

A bit of history...

The well-posedness of the weakly hyperbolic equations has been a challenging problem for a long time.

Example

Second order Cauchy problem in one dimension

$$\begin{aligned} \partial_t^2 u - a(t,x)\partial_x^2 u &= 0, \\ u(0,x) &= g_1(x), \\ \partial_t u(0,x) &= g_2(x). \end{aligned} \tag{1}$$

Up until now there is no characterisation of smooth functions $a(t,x) \geq 0$ for which (1) would be C^∞ well-posed.

Sufficient conditions

There are sufficient conditions: Oleinik 1970 (*Comm. Pure Appl. Math*) (1) is C^∞ well-posed provided there is a constant $C > 0$ such that

$$Ca(t,x) + \partial_t a(t,x) \geq 0.$$

In case of $a(t,x) = a(t)$ Oleinik's condition is satisfied for $a(t) \geq 0$ with $a'(t) \geq 0$.

No C^∞ well-posedness

If we drop Oleinik's condition we have examples of no C^∞ well-posedness. Colombini-Spagnolo 1982 (*Acta*): there exist $a \in C^\infty([0,T])$, $a(t) \geq 0$ and $g_1, g_2 \in C^\infty(\mathbb{R})$ such that (1) has no sol. in $C^1([0,T], \mathcal{D}'(\mathbb{R}_x))$.

Usually well-posedness in obtained in Gevrey classes:

$$\mathcal{A}(\mathbb{R}^n) \subseteq G^s(\mathbb{R}^n) \subseteq C^\infty(\mathbb{R}^n).$$

A bit of history of weakly hyperbolic problems...

Second order equations with t-dependent coefficients

- Nishitani 1980 (*Comm. Part. Diff. Eq.*)
- Colombini, Jannelli and Spagnolo 1987 (*Ann. of Math*)

Higher order equations with t-dependent coefficients

- Colombini and Kinoshita 2002 (*JDE*): m-order, Hölder roots
- Kinoshita and Spagnolo 2006 (*Math. Ann.*): hom. equations, non-analytic coeff.
- Jannelli and Taglialatela 2011 (*JDE*): hom. equations, analytic coeff.

Equations with (t, x)-dependent coefficients

- Bronshtein 1982: Gevrey dependence in x,
- Ohya and Tarama 1984: Hölder in t, Gevrey in x

My research on t-dependent scalar equations

- G and Ruzhansky 2012 (JDE): *On the well-posedness of weakly hyperbolic equations with time dependent coefficients*
- G and Ruzhansky 2013 (Math. Ann.): *Weakly hyperbolic equations with non-analytic coefficients and lower order terms*
- G and Ruzhansky 2015 (JDE): *Wave equation for sums of squares on compact Lie groups*

My research on t-dependent systems

- G and Jäh 2017 (Math. Ann.): *Well-posedness of hyperbolic systems with multiplicities and smooth coefficients*
- G and Ruzhansky 2017 (Ann. Mat. Pura Appl.): *On C^∞ well-posedness of hyperbolic systems with multiplicities*

My research on irregular coefficients

- G and Ruzhansky 2015 (Arch. Rat. Mech. Anal): *Hyperbolic second order equations with non-regular time dependent coefficients*

The (t,x)-dependent case: joint with Jäh and Ruzhansky (2018)

We are studying

$$\begin{cases} D_t u = A(t,x,D_x)u + B(t,x,D_x)u + f(t,x), & (t,x) \in [0,T] \times \mathbb{R}^n, \\ u|_{t=0} = u_0, & x \in \mathbb{R}^n, \end{cases}$$

where $A(t,x,D_x)$ is a hyperbolic $m \times m$-matrix with entries in $C([0,1], \Psi^1_{1,0}(\mathbb{R}^n))$ and $B(t,x,D_x)$ has entries in $C([0,1], \Psi^0_{1,0}(\mathbb{R}^n))$.

In 2013, Gramchev and Ruzhansky obtained Sobolev well-posedness and C^∞ well-posedness when $m = 2$ under a certain hypothesis on the eigenvectors $\Rightarrow$ Reduction to upper triangular form

Extension to any matrix size via upper-triangularisation and FIO methods

Toy model: 2×2

Let

$$\begin{cases} D_t u = A(t,x,D_x)u + B(t,x,D_x)u + f(t,x), & (t,x) \in [0,T] \times \mathbb{R}^n, \\ u|_{t=0} = u_0, & x \in \mathbb{R}^n, \end{cases}$$

where $u_0(x) = \left(u_1^0(x), u_2^0(x)\right)^T$, $f(t,x) = \left(f_1(t,x), f_2(t,x)\right)^T$, and with

$$A(t,x,D_x) = \begin{pmatrix} \lambda_1(t,x,D_x) & a_{12}(t,x,D_x) \\ 0 & \lambda_2(t,x,D_x) \end{pmatrix}$$

and

$$B(t,x,D_x) = \begin{pmatrix} b_{11}(t,x,D_x) & b_{12}(t,x,D_x) \\ b_{21}(t,x,D_x) & b_{22}(t,x,D_x) \end{pmatrix}.$$

Needed background on FIO's

For each eigenvalue λ_j of A, we denote by $G_j^0\theta$ the solution to

$$\begin{cases} D_t w = \lambda_j(t,x,D_x)w + b_{jj}(t,x,D_x)w, \\ w(0,x) = \theta(x), \end{cases}$$

and by $G_j g$ the solution to

$$\begin{cases} D_t w = \lambda_j(t,x,D_x)w + b_{jj}(t,x,D_x)w + g(t,x), \\ w(0,x) = 0. \end{cases}$$

The operators G_j^0 and G_j can be microlocally represented by Fourier integral operators

$$G_j^0\theta(t,x) = \int\limits_{\mathbb{R}^n} e^{\mathrm{i}\varphi_j(t,x,\xi)} a_j(t,x,\xi)\widehat{\theta}(\xi)\,d\xi$$

and

$$G_j g(t,x) = \int\limits_0^t \int\limits_{\mathbb{R}^n} e^{\mathrm{i}\varphi_j(t,s,x,\xi)} A_j(t,s,x,\xi)\widehat{g}(s,\xi)\,d\xi\,ds.$$

The phase function $\varphi_j(t, s, x, \xi)$ solves the eikonal equation

$$\begin{cases} \partial_t \varphi_j = \lambda_j(t, x, \nabla_x \varphi_j), \\ \varphi_j(s, s, x, \xi) = x \cdot \xi, \end{cases}$$

and we use the notation

$$\varphi_j(t, x, \xi) = \varphi_j(t, 0, x, \xi).$$

Here we also have the amplitudes $A_{j,-k}(t, x, \xi)$ of order $-k$, $k \in \mathbb{N}$, giving $A_j \sim \sum_{k=0}^{\infty} A_{j,-k}$, and they satisfy the transport equations with initial data at $t = s$, and we have $a_j(t, x, \xi) = A_j(t, 0, x, \xi)$.
We will make use of the following mapping properties:

For any $\sigma \in \mathbb{R}$, for sufficiently small t, we have

$$\left\| G_j^0 \theta(t) \right\|_{H^\sigma} \leq C \|\theta\|_{H^\sigma}, \quad \|G_j g(t)\|_{H^\sigma} \leq Ct \|g\|_{L_s^\infty H_x^\sigma}.$$

In this specific 2×2 case we have

$$
\begin{aligned}
u_1 &= U_1^0 + G_1((a_{12} + b_{12})u_2), && (2)\\
u_2 &= U_2^0 + G_2(b_{21}u_1), && (3)
\end{aligned}
$$

where

$$U_j^0 = G_j^0 u_j^0 + G_j(f_j), \quad j = 1, 2. \tag{4}$$

Plugging (3) in (2), we obtain

$$u_1 = \tilde{U}_1^0 + G_1(a_{12}G_2(b_{21}u_1)) + G_1(b_{12}G_2(b_{21}u_1)), \tag{5}$$

where

$$\tilde{U}_1^0 = G_1^0 u_1^0 + G_1(f_1) + G_1((a_{12} + b_{12})U_2^0). \tag{6}$$

- The operator $G_1 \circ a_{12} \circ G_2 \circ b_{21}$ in (5) acts continuously on H^s if it is of order 0. Since $a_{12} \in C\Psi^1_{1,0}$ we therefore need to assume that $b_{21} \in C\Psi^{-1}_{1,0}$.
- The operator $G_1 \circ b_{12} \circ G_2 \circ b_{21}$ belongs to $CI^{-1}_{1,0}$ since $b_{21} \in C\Psi^{-1}_{1,0}$ and $b_{12} \in C\Psi^0_{1,0}$.
- Banach spaces $X(t) := C([0, T], H^s)$, $t \in [0, T]$, equipped with the norm $\|u\|_{X(t)} = \sup_{\tau \in [0,t]} \|u(\tau, \cdot)\|_{H^s}$.

Let

$$\mathcal{G}^0_1 u_1 := G_1(a_{12} G_2(b_{21} u_1)) + G_1(b_{12} G_2(b_{21} u_1)).$$

It follows that (5) can be written as

$$u_1 = \tilde{U}^0_1 + \mathcal{G}^0_1 u_1.$$

- $\mathcal{G}^0_1$ is a **contraction**: there exists $T^* \in [0, T]$ such that

$$\|\mathcal{G}^0_1(u - v)\|_{X(t)} \leq C_{a,s} T^* \|u - v\|_{X(t)},$$

with $C_{a,s} T^* < 1$.

Conclusion:

- Banach's fixed point theorem ensures the existence of a unique fixed point u_1 for the map $\mathcal{G}_1^0$.
- By assuming that the initial data $\tilde{U}_1^0$ belongs to $C([0,T^*],H^s)$ we conclude that there exists a unique $u_1 \in C([0,T^*],H^s)$ solving (5).
- Note that the same argument proves that the operator $I - \mathcal{G}_1^0$ is invertible on a sufficiently small interval in t since $\mathcal{G}_1^0 = I$ at $t = 0$.
- From formula (6) it is clear that in order to get $\tilde{U}_1^0$ to belong to $C([0,T^*],H^s)$ we need to assume that $U_2^0 \in H^{s+1}$ (and therefore $f_2 \in H^{s+1}$).
- Finally, we get u_2 by substitution of u_1 in (3).

General result: G., Jäh, Ruzhansky, Math. Annalen, 2018

Let

$$\begin{cases} D_t u = A(t,x,D_x)u + B(t,x,D_x)u + f(t,x), & (t,x) \in [0,T] \times \mathbb{R}^n, \\ u|_{t=0} = u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where $A(t,x,D_x) = (a_{ij}(t,x,D_x))_{i,j=1}^m$ is an upper-triangular matrix of pseudodifferential operators of order 1 with real eigenevalues, and $B(t,x,D_x) = (b_{ij}(t,x,D_x))_{i,j=1}^m$ is a matrix of pseudo-differential operators of order 0, continuous with respect to t.
Hence, if the lower order terms b_{ij} belong to $C([0,T], \Psi^{j-i})$ for $i > j$, $u_k^0 \in H^{s+k-1}$ and $f_k \in C([0,T], H^{s+k-1})$ for $k = 1, \dots, m$ then the problem above has a unique anisotropic Sobolev solution u, i.e., $u_k \in C([0,T], H^{s+k-1})$ for $k = 1, \dots, m$.

Recent results when the matrix of the principal part A is t-independent

$$\begin{cases} D_t u = A(x, D_x)u + B(t, x, D_x)u + f(t, x), & (t, x) \in [0, T] \times \mathbb{R}^n, \\ u\big|_{t=0} = u_0(x), & x \in \mathbb{R}^n. \end{cases}$$

Hypotheses:

(H1) (**Lower order terms**)

$$b_{ij} \in C([0, T], \Psi^{j-i}) \quad \text{for } i > j;$$

(H2) (**multiplicities**) (Kamotski and Ruzhansky, 2007):
$\exists M \in \mathbb{N}$ such that if $\lambda_j(x, \xi) = \lambda_k(x, \xi)$ for some $j, k \in \{1, \dots, m\}$ and $\lambda_j(x, \xi) \not\equiv \lambda_k(x, \xi)$ near (x, ξ) then $\exists N \leq M$ such that

$$\lambda_j(x, \xi) = \lambda_k(x, \xi) \;\Rightarrow\; H^N_{\lambda_j}(\lambda_k) := \{\lambda_j, \{\lambda_j, \dots, \{\lambda_j, \lambda_k\}\} \dots\} \neq 0.$$

G, Jäh and Ruzhansky, 2020

Let $A(x, D_x)$ is an upper-triangular matrix of pseudo-differential operators of order 1 and $B(t, x, D_x)$ is a matrix of pseudo-differential operators of order 0, continuous with respect to t. Let $u_j^0 \in H^{s+j-1}(\mathbb{R}^n)$ and $f_j \in C([0, T], H^{s+j-1})$ for $j = 1, \ldots, m$. Hence, under (H1) and (H2),

(i) the Cauchy problem above has a unique anisotropic Sobolev solution u, i.e., $u_j \in C([0, T], H^{s+j-1})$ for $k = 1, \ldots, m$;

(ii) for any $N \in \mathbb{N}$, the components u_j, $j = 1, \ldots, m$ of the solution u are given by

$$u_j(t, x) = \sum_{l=1}^{m} \left(\mathcal{H}_{j,l}^{l-j}(t) + R_{j,l}(t) \right) u_l^0 + \left(\mathcal{K}_{j,l}^{l-j}(t) + S_{j,l}(t) \right) f_l$$

where $R_{j,l}$, $S_{j,l} \in \mathcal{L}(H^s, C([0, T], H^{s+N-l+j}))$ and the operators $\mathcal{H}_{j,l}^{l-j}$, $\mathcal{K}_{j,l}^{l-j} \in \mathcal{L}(C([0, T], H^s), C([0, T], H^{s-l+j}))$ are integrated Fourier Integral Operators of order $l - j$.

Propagation of singularities

Under our hypotheses

$$WF(u_j(t,\cdot)) \subset \left(\bigcup_{l=1}^{m} WF(\mathcal{H}_{j,l}^{l-j}(t)u_l^0)\right) \bigcup \left(\bigcup_{l=1}^{m} WF(\mathcal{K}_{j,l}^{l-j}(t)f_l)\right),$$

with each of the wave front sets for terms in the right hand side given by the propagation along the broken Hamiltonian flow.

Broken Hamiltonian flow

It means that points follow bicharacteristics of λ_{j_1} until meeting the characteristic of λ_{j_2}, and then continue along the bicharacteristic of λ_{j_2}, etc.

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