Degenerate Elliptic Boundary Value Problems with Non-smooth Coefficients

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The Situation

X smooth n-dimensional manifold with boundary, of bounded geometry
A strongly elliptic second order operator, locally

$$A = -\sum_{j,k=1}^{n} a^{jk}(x)\partial_{x_j}\partial_{x_k} + \sum_{j=1}^{n} b^j(x)\partial_{x_j} + c(x).$$

Assumptions on A

1. $a^{jk} = a^{kj}$ real-valued $C^\tau(X)$, $\tau > 0$, b^j, c in $L^\infty(X)$.
2. A is strongly elliptic: There exist constants $\Theta_0, \theta_0 > 0$ such that

$$\theta_0|\xi|^2 \leq \sum_{i,j=1}^{n} a^{jk}(x)\xi_i\xi_j \leq \Theta_0|\xi|^2,$$

The Situation

$$A = -\sum_{i,j=1}^{n} a^{ij}(x)\partial_{x_i}\partial_{x_j} + \sum_{j=1}^{n} b^j(x)\partial_{x_j} + c(x).$$

Boundary Condition

$$Tu = \mu_0\gamma_0 u + \mu_1\gamma_1 u$$

- $\mu_0, \mu_1 \in C^\infty(\partial X)$, $\mu_0, \mu_1 \geq 0$, $\mu_0(x) + \mu_1(x) > 0$ for all x on ∂X.
- $\gamma_0 u = u|_{\partial X}$, $\gamma_1 u = \partial_\nu u|_{\partial X}$ exterior normal derivative at ∂X.

Boundary Value Problem

$$Au = f \text{ in } X, \quad Tu = \varphi \text{ on } \partial X$$

for given functions f on X and φ on ∂X.

The Situation

$$\begin{aligned} A &= -\sum_{i,j=1}^{n} a^{ij}(x)\partial_{x_i}\partial_{x_j} + \sum_{j=1}^{n} b^j(x)\partial_{x_j} + c(x), \\ T &= \mu_0\gamma_0 + \mu_1\gamma_1. \end{aligned}$$

Clearly:

- $\mu_0 \equiv 1$, $\mu_1 \equiv 0$: Dirichlet Problem
- $\mu_0 \equiv 0$, $\mu_1 \equiv 1$: Neumann Problem
- $\mu_1 > 0$: Robin Problem
- Not elliptic, unless $\mu_1 \equiv 0$ or $\mu_1 > 0$ (order changes).

Note

Studied extensively by K. Taira with C^∞ coefficients
Describes certain stochastic diffusion processes
Related to degenerate oblique derivative problem (Poincaré 1910)
My interest: Quasilinear pde involving (A, T)
Stefan Problem on inhomogeneous background – not yet.

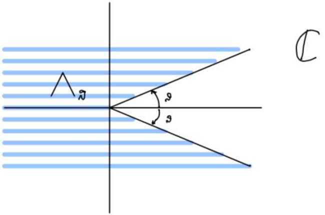
What is Known

Consider the realization A_T, i.e. the operator A, acting on the domain

$$\mathscr{D}(A_T) = \{u \in H^2_p(X) : Tu = 0\}.$$

For $\theta > 0$ define the sector in $\mathbb{C}$

$$\Lambda_\theta = \{\lambda = re^{i\phi} : r \geq 0;\ |\phi| \geq \theta\}.$$



Theorem (Taira 1995)

If the coefficients of A and μ_0, μ_1 are smooth, then A_T generates an analytic semigroup on Λ_θ for each $\theta > 0$.

Main Results

Recall

$$\mathscr{D}(A_T) = \{u \in H^2_p(X) : Tu = 0\}.$$

$$\Lambda_\theta = \{\lambda = re^{i\phi} : r \geq 0;\ |\phi| \geq \theta\}.$$

Sectoriality

We find $C, R > 0$ such that $\lambda - A_T$ is invertible for $\lambda \in \Lambda_\theta$, $|\lambda| > R$, and

$$\|\lambda(\lambda - A_T)^{-1}\|_{\mathscr{L}(L^p(X))} \leq C.$$

Bounded H_∞-calculus

For any $\theta > 0$ there exists a $c > 0$ such that the operator $c - A_T$ has a bounded H_∞-calculus on $\mathbb{C} \setminus \Lambda_\theta$.

Application: Short Time Solution to the PME

$\partial_t v - \Delta v^m = 0,\ Tv = \phi,\ v(0) = v_0 > 0$

Background: Bounded H_∞-calculus

McIntosh 1986: Let B be an unbounded operator on $L^p(X)$ such that

$$\sup_{\lambda \in \Lambda_\theta} \|\lambda(B-\lambda)^{-1}\|_{\mathscr{L}(L^p(X))} < +\infty.$$

For a bounded holomorphic (H_∞) function f on $\mathbb{C} \setminus \Lambda$ define $f(B)$ by

$$f(B) = \frac{1}{2\pi i} \int_{\partial \Lambda} f(\lambda)(\lambda - B)^{-1}\, d\lambda.$$

B admits a bounded H_∞-calculus, if there exists a C such that

$$\|f(B)\|_{\mathscr{L}(L^p(X))} \leq C\|f\|_\infty.$$

Theorem (Dore and Venni)

Provided the angle θ is $< \pi/2$, the existence of a bounded H_∞-calculus implies maximal regularity. Allows to solve quasilinear problems $\partial_t u + B(u)u = f(t,u)$, $u(0) = u_0$ via a theorem by Clément and Li.

Strategy

- First consider the case, where $X = \mathbb{R}^n_+$ and coefficients are C^∞_b.
- Write $\lambda = e^{i\phi}\mu^2$ for $\mu \geq 0$, $|\phi| \geq \theta/2$.
- Construct a good parameter-dependent parametrix to $e^{i\phi}\mu^2 - A_T$. This will allow to get the H_∞-estimates for the smooth case.
- For this use Agmon's trick: Replace $e^{i\phi}\mu^2$ by $e^{i\phi}D_z^2$ for a variable $z \in \mathbb{R}$ and work on $X \times \mathbb{R}$. Solving the parameter-dependent problem reduces to standard problem.
- Reduce the problem to the boundary. Involves the operator $S = \mu_1\Pi + \mu_0$ where Π is the Dirichlet-Neumann operator. It is only hypoelliptic and has a parametrix in $S^0_{1,1/2}$.
- Construct the full parametrix using a Boutet de Monvel calculus with symbols of $S_{1,\delta}$-type.
- Apply H_∞-perturbation results for non-smooth coefficients.
- Extend to manifolds of bounded geometry with boundary.
- Obtain solution to porous medium eq'n via Clément-Li's theorem.

Steps in the Strategy

We have to show

$$\|f(A_T)\|_{\mathscr{L}(L^p(X))} \leq C\|f\|_\infty$$

for $f(A_T)$ defined by the integral:

$$f(A_T) = \frac{1}{2\pi i}\int_{\partial\Lambda} f(\lambda)(\lambda - A_T)^{-1}\, d\lambda.$$

- Instead of $(\lambda - A_T)^{-1}$ consider the principal term in the parametrix.
- Only for this term we have to establish the estimate.
- All other terms (and the difference between the parametrix and the resolvent) decay faster than λ^{-1}. For these, the estimate is trivial.

Agmon's Trick

Let $A_\phi = A + e^{i\phi} D_z^2$, so $\sigma(A_\phi) = \sigma(A) + e^{i\phi}\zeta^2 \in S^2_{1,0}$.
Solving the problem for A_ϕ is equivalent to solving the parameter-dependent problem for A.
Well-known: In the smooth case, the Dirichlet problem has a parametrix in Boutet de Monvel's calculus.

$$\begin{pmatrix} A_\phi \\ \gamma_0 \end{pmatrix}^{-\#} = \begin{pmatrix} A_{\phi,+}^{-\#} + G_\phi^D & K_\phi^D \end{pmatrix}.$$

where $A_\phi^{-\#}$ is a parametrix to A and G_ϕ^D is a singular Green operator and K_ϕ^D is a Poisson type operator.

Reduction to the Boundary

Recall:

$$\begin{pmatrix} A_\phi \\ \gamma_0 \end{pmatrix}^{-\#} = \begin{pmatrix} A_{\phi,+}^{-\#} + G_\phi^D & K_\phi^D \end{pmatrix}.$$

Since $TK_\phi^D = (\mu_1\gamma_1 + \mu_0\gamma_0)K_\phi^D = \mu_1\Pi_\phi + \mu_0$ we find

$$\begin{pmatrix} A_\phi \\ T \end{pmatrix}\begin{pmatrix} A_{\phi,+}^{-\#} + G_\phi^D & K_\phi^D \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ T(A_{\phi,+}^{-\#} + G_\phi^D) & \underbrace{\mu_1\Pi_\phi + \mu_0}_{S_\phi}, \end{pmatrix}.$$

where $\Pi_\phi = \gamma_1 K_\phi^D$ is the Dirichlet to Neumann operator.

$\Rightarrow$ A parametrix $S_\phi^{-\#}$ to $S_\phi = \mu_1\Pi_\phi + \mu_0$ gives one to $\binom{A_\phi}{T}$.

Problem

$S_\phi = \mu_1\Pi + \mu_0$ is not elliptic.

But: Hypoelliptic and has a parametrix $S_\phi^{-\#}$ in $S^0_{1,1/2}$ (loss of 1 order).

Parametrix Construction

$$\begin{pmatrix} A_\phi \\ T \end{pmatrix}^{-\#} \sim (A_{\phi,+}^{-\#} + G_\phi^D - K_\phi^D S_\phi^{-\#} T(A_{\phi,+}^{-\#} + G_\phi^D) \quad K_\phi^D S_\phi^{-\#})$$

$$\text{so } (A_{\phi,T})^{-\#} \sim A_{\phi,+}^{-\#} + G_\phi^D - K_\phi^D S_\phi^{-\#} T(A_{\phi,+}^{-\#} + G_\phi^D)$$

We want to show that the red term is in an extended version of Boutet de Monvel's calculus.
The various operator classes in Boutet de Monvel's calculus can be viewed as operator-valued pseudodifferential operators.
A potential operator K or a singular Green operator G of class zero for example have symbols

$$k(x', \xi') \in S^\mu(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathbb{C}, \mathscr{S}(\mathbb{R}_+))$$
$$g(x', \xi) \in S^\mu(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathscr{S}'(\mathbb{R}_+), \mathscr{S}(\mathbb{R}_+))$$

Replace S^μ by $S^\mu_{1,\delta}$, make sure, calculus works.

Strategy

Consider $K^D_\phi S^{-\#}_\phi T(A^{-\#}_{\phi,+} + G^D_\phi)$ in extended version of BdM's calculus.

- Note $\gamma_0(A^{-\#}_{\phi,+} + G^D_\phi) = 0$ by definition. So the operator in red is $K^D_\phi S^{-\#}_\phi \mu_1\gamma_1(A^{-\#}_{\phi,+} + G^D_\phi)$
- Have explicit formulas for the principal symbols g^D_ϕ of G^D_ϕ and k^D_ϕ of K^D_ϕ in Boutet de Monvel's calculus.
- Problem: Loss of order: S_ϕ had order 1, the parametrix order 0.
- But: The symbol of $S^{-\#}_\phi \mu_1$ is in $S^{-1}_{1,1/2}$. Recover the order.
- With this show that the red term is in fact an extended singular Green symbol of order -2.
- With the concrete formula, we can estimate the integral term for $f(A_T)$.

Nonsmooth Coefficients: H_∞-perturbation Results

Let A have a bounded H_∞-calculus in the UMD Banach space E and $0 \in \rho(A)$. Suppose that B is a linear operator in E with $\mathscr{D}(B) \supseteq \mathscr{D}(A)$.

Theorem. Same order perturbations. Let $\gamma \in (0,1)$, $C \geq 0$, with

$$B(\mathscr{D}(A^{1+\gamma})) \subseteq \mathscr{D}(A^\gamma) \text{ and } \|A^\gamma Bu\|_E \leq C\|A^{1+\gamma}u\|_E, \quad u \in \mathscr{D}(A^{1+\gamma}).$$

Then $A + B$ has a bounded H_∞-calculus, provided $\|Bu\|_E \leq \varepsilon\|Au\|_E$ for suitably small $\varepsilon > 0$.

Theorem. Lower order perturbations. Assume $\gamma \in (0,1)$, $C \geq 0$ with

$$\|Bu\|_E \leq C\|A^{1-\gamma}u\|_E, \quad u \in \mathscr{D}(A).$$

Then $c + A + B$ has a bounded H_∞-calculus for c sufficiently large.

Low Regularity

- Apply freezing-of-coefficients to the principal part and treat all lower order terms as perturbations.
- Work on a lattice in $\mathbb{R}^n_+$ and identify operators on this lattice by operators on sequence spaces.
- Similarly for manifolds of bounded geometry.

Application to PME

Choose p, q such that $n/p + 2/q < 1$. Let $v_0 \in H_p^2(X)$ be strictly positive. Transform: $v = u + v_0$ and instead of

$$\dot{v} - \Delta v^m = 0; \, Tv = \phi; \, v(0) = v_0$$

(with $\phi = Tv_0$ for compatibility) study

$$\begin{aligned} \dot{u} - \Delta(u + v_0)^m &= 0; \\ Tu &= 0; \\ u(0) &= 0. \end{aligned}$$

which falls in our setting. Show that the conditions in the theorem of Clément and Li are satisfied. Obtain solution

$$u \in L^q(0, T; H_p^2(X) \cap \ker T) \cap W_q^1(0, T; L^p(X)).$$

Hence the corresponding result is true for v.

Thank You for Your Attention

Manifolds with Boundary of Bounded Geometry (Ammann, Große, Nistor)

A riemannian manifold (M, g) has bounded geometry, if it has positive injectivity radius and all covariant derivatives of the curvature tensor are bounded.
$(X, g_{|X})$ manifold with boundary and bounded geometry in (M, g), if

- (X, g) isometrically embedded subset of the same dimension
- ∂X closed subset of M with globally defined outward unit normal vector filed
- $(\partial X, g_{|\partial X})$ manifold with bounded geometry
- The second fundamental form Π of ∂X in M and all its covariant derivatives are bounded
- There exists $\delta > 0$ such that $\exp^\perp : \partial X \times (-\delta, \delta) \to M$ is injective.

Background: Maximal Regularity and Evolution Equations

In order to solve a quasilinear parabolic equation

$$\dot{u} + A(u)u = g(t,u),\ 0 < t < T_0, \quad u|_{t=0} = u_0, \tag{1}$$

we may apply:

Theorem. Clément-Li, 1993.

Let $X_1 \hookrightarrow X_0$ be Banach spaces, $1 < q < \infty$. (1) has short-time solution

$$u \in L^q((0,T_1),X_1) \cap W^1_q((0,T_1),X_0),\ T_1 \le T_0,$$

whenever there exists a neighborhood U of u_0 in $X_q = (X_1, X_0)_{1/q,q}$ with

(H1) $u \mapsto A(u) \in Lip(U; \mathscr{L}(X_1, X_0))$,

(H2) $(t,u) \mapsto g(t,u) \in Lip((0,T_0) \times U, X_0)$, and

(H3) $A(u_0)$ has maximal regularity.

In that case, $u \in C([0,T_1], X_q)$.

Background

A Hierarchy of Properties

- Bounded H_∞-calculus* $\Rightarrow$
- Bounded imaginary powers* $\Rightarrow$
- R-sectoriality $\Rightarrow$
- Maximal regularity $\Rightarrow$
- Generation of analytic semigroup

Also of interest

The existence of bounded imaginary powers implies that

$$[\mathcal{D}(A^\alpha), \mathcal{D}(A^\beta)]_\theta = \mathcal{D}(A^{(1-\theta)\alpha+\theta\beta}))$$

and precise mapping properties of fractional powers.

* For large angles