Bergman kernels in the analytic case

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Advances in pseudo-differential operators,
Ghent, July 7-8, 2020

0. Introduction

The subject is very much related to Fourier integral operators with complex phase and non-self-adjoint operators, Hörmander [Ho60a, Ho60b], Sj (thesis) [Sj73], J.J. Duistermaat–Sj [DuSj73], Melin–Sj [MeSj74], L. Boutet de Monvel [Bo74] with an appendix about complexes of pseudodifferential operators. (Cf. [SaKaKa73].) and C. Fefferman [Fe74], Boutet–Sj [BoSj75] for the singularity at the diagonal of the boundary of the Bergman kernel for strictly pseudoconvex domains.

In analytic microlocal analysis [Sj82], [HiSj18], resonances [HeSj86] and non-self-adjoint spectral asymptotics [MeSj02, MeSj03], Hitrik–Sj [HiSj18b], it is natural to work in exponentially weighted spaces of holomorphic functions, using approximations of the corresponding Bergman kernel.

New interest in the Bergman kernel through the works of Tian [Ti90], T. Bouche [Bo90], Catlin [Ca99], Zelditch [Ze98] and others. The object of study was then high powers of a complex line bundle with positive curvature, on a compact complex manifold. Catlin and Zelditch derived a full asymptotic formula by applying [BoSj75] to a suitable domain. Locally the problem is equivalent to one for weighted spaces of holomorphic functions, and with Berndtsson and Berman [BeBeSj08] we got a self-contained proof of the asymptotics in the case of line bundles.

More recently there have been works in the case of real-analytic line bundles, with exponentially small remainders [HeLuXu17], [RoSjVu18], A. Deleporte [De18], L. Charles [Ch19], [HeXu19]. In [DeHiSj20] we obtained a further simplification. There is also a very elegant approach by Kashiwara [Ka77] in the case of domains with analytic boundary.

1. Strictly pseudoconvex domains and Fourier integral operators with complex phase

Let P be a pseudodifferential operator on a smooth compact manifold X with principal symbol $p(x,\xi) \in C^\infty(T^*X)$. Assume that $i^{-1}\{p,\overline{p}\} \neq 0$ on $\Sigma := p^{-1}(0) \cap (T^*X \setminus 0)$. Let $\Sigma_\pm = \{\rho \in \Sigma;\ \pm i^{-1}\{p,\overline{p}\}(\rho) > 0\}$.
[DuSj73]: $\exists$ operators F, F_+, $F_- : \mathcal{D}'(X) \to \mathcal{D}'(X)$ that do not spread singularities, with $F_\pm$ concentrated to $\Sigma_\pm$ such that modulo smoothing operators:

$$F_+ + FP \equiv 1,\ F_- + PF \equiv 1,\ F_\pm^* \equiv F_\pm.$$

$A^* =$ the adjoint of A in $L^2(X, dx)$ for some positive smooth density dx. We can say that "F_+, F_- are microlocal orthogonal projections onto $\mathcal{N}(P)$ and $\mathcal{N}(P^*)$ respectively".

[MeSj74]: $F_\pm$ are Fourier integral operators with complex phase.

Let $\Omega \Subset \mathbf{C}^n$, $n \geq 2$ be a strictly pseudoconvex domain with smooth boundary, X. We then have the $\overline{\partial}$-complex on Ω and the induced $\overline{\partial}_b$-complex on X. The Bergman projection B is the orthogonal projection onto the kernel of $\overline{\partial}$ on the level of forms of degree 0 (i.e. scalar functions). The Szegő projection S is the corresponding object for $\overline{\partial}_b$. The structure of $\overline{\partial}_b$ generalizes the one of the scalar operator P above. F_+ becomes the Szegő projection.
Let Ω be given by $\Phi < 0$ where Φ is smooth and $d\Phi \neq 0$ on X. There exists $\Psi(x, y) \in C^\infty(\mathbf{C}^n \times \mathbf{C}^n)$ such that $\Psi(x, \overline{x}) = \Phi(x)$ and

$$\overline{\partial}_{x,y}\Psi(x, y) \text{ vanishes to } \infty \text{ order on } \{(x, \overline{x}); x \in \mathbf{C}^n\}.$$

We showed in [BoSj75] that S is a Fourier integral operator with complex phase: If $K_S(x, y)$ denotes the distribution kernel of S, then

$$K_S(x, y) \equiv \int_0^{+\infty} e^{it\Psi(x,\overline{y})} s(x, \overline{y}; t)dt, \; x, y \in X,$$

where

$$s \sim \sum_{k=0}^{\infty} t^{n-1-k} s_k(x, \overline{y}; t) \text{ in } C^\infty.$$

There is a similar formula for the distribution kernel K_B of B, now for $x, y \in \Omega$ and with an amplitude

$$b \sim \sum_{k=0}^{\infty} t^{n-k} b_k(x, \overline{y}; t).$$

2. Powers of line bundles, weighted spaces of holomorphic functions

Let $\mathcal{L}$ be a complex line bundle over a complex compact manifold X, of dimension n. Assume $\mathcal{L}$ is equipped with a metric. We can define a curvature form, namely the $(1,1)$-form $\partial_z \overline{\partial}_z \Phi$, where locally $|s| = e^{\Phi}$ for some non-vanishing holomorphic section s. We assume that this curvature is strictly positive. Let $\mathcal{L}^k = \mathcal{L} \otimes \ldots \otimes \mathcal{L}$. The Bergman projection is the orthogonal projection $\Pi_k : L^2(X; \mathcal{L}^k) \to (L^2 \cap \mathrm{Hol})(X; \mathcal{L}^k)$. Catlin and Zelditch gave a complete asymptotic expansion for Π_k as $k \to +\infty$. Locally, we take a section s as above and represent general sections of $\mathcal{L}^k$ as $u\, s^k$. The problem is then to study the orthogonal projection (also denoted) Π_k:

$$\underbrace{L^2(\Omega, e^{-2k\Phi} L(dx))}_{L^2_\Phi(\Omega)} \to \underbrace{L^2(\Omega, e^{-2k\Phi} L(dx)) \cap \mathrm{Hol}\,(\Omega)}_{H_\Phi(\Omega)}.$$

Here for simplicity we assume that $L(dx)$ is the Lebesgue measure.

Theorem (Catlin, Zelditch)

In the local representation above, we have

$$\Pi_k u(x) = \int e^{2k\Psi(x,\overline{y})} a(x,\overline{y};k) u(y) e^{-2k\Phi(y)} L(dy), \tag{1}$$

where $a \sim \sum_0^{+\infty} k^{n-j} a_j(x,\overline{y})$ *in* C^∞, *where* a_j *and* Ψ *are holomorphic to* ∞*-order on the anti-diagonal and* $\Psi(x,\overline{x}) = \Phi(x)$.

By Taylor expansion, we know that near the diagonal

$$-\Phi(x) + 2\Re\Psi(x,\overline{y}) - \Phi(y) \asymp -|x-y|^2,$$

which implies nice continuity properties for Π_k. The proofs of Catlin and Zelditch used a reduction to the main result of [BoSj75].

Here are some ideas from the direct proof in [BeBeSj08]: Semi-classical pseudodifferential operators in H_Φ-spaces, with $h = 1/k$, take the form ([Sj82]):

$$Bu(x) = \frac{1}{(2\pi h)^n} \iint e^{\frac{i}{h}(x-y)\cdot\eta} b(x, y, \eta; h)u(y)dyd\eta, \tag{2}$$

where the symbol b is holomorphic or almost holomorphic and we can use the "good contour"

$$\theta = \frac{2}{i}\partial_x\Phi\left(\frac{x+y}{2}\right) + \frac{i}{C}\overline{(x-y)}.$$

When $b = 1$ this gives the identity operator (up to small errors). Try to express B with a non-standard phase:

$$Bu(x) = h^{-n} \iint e^{\frac{2}{h}(\Psi(x,\theta)-\Psi(y,\theta))} a(x, y, \theta; h)u(y)dyd\theta \tag{3}$$

and notice that with the contour $\theta = \overline{y}$, we get

$$Bu(x) = h^{-n} \iint e^{\frac{2}{h}(\Psi(x,\overline{y})-\Phi(y))} a(x, y, \overline{y}; h)u(y)dyd\overline{y} \tag{4}$$

The Kuranishi trick shows that (2) and (3) are equivalent: By Taylor expansion, $\Psi(x,\theta)-\Psi(y,\theta)=(x-y)\cdot\eta(x,y,\theta)$ and we can pass from θ to η by a change of variables. The formula (4) is almost what we want for Π, but we still need to eliminate the y-dependence in a. This can be done by a procedure of divisions and integrations by parts (allowing to gain a power of h at each appearance of a factor x_j-y_j). After that, it is fairly straight forward to see that the resulting operator is an approximation of the Bergman projection. The full proof requires more work of course.

3. The analytic case

The precise meaning of (1) is that if we introduce the distribution kernel of Π_k through

$$\Pi_k u(x) = \int K_{\Pi_k}(x,\overline{y})u(y)e^{-2\Phi(y)/h}L(dy),\ \ h=1/k,$$

then on the level of effective kernels (i.e. for the operator $e^{-\Phi/h}\circ\Pi_k\circ e^{\Phi/h}$), we have

$$\begin{aligned} e^{-\Phi(x)/h}\left(K_{\Pi_k}(x,\overline{y}) - e^{2\Psi(x,\overline{y})/h}a(x,\overline{y};h)\right)e^{-\Phi(y)/h} \\ = \mathcal{O}(h^\infty). \end{aligned} \tag{5}$$

Assume now that the metric on $\mathcal{L}$ is real analytic. Then in the local description, Φ becomes real-analytic and we can choose Ψ holomorphic. We then expect the symbol in (1) to be a classical analytic symbol in the sense of Boutet de Monvel and Krée [BoKr67] and the remainder in (5) to be exponentially small.

Classical analytic symbols. Let $V \subset \mathbf{C}^n$ be open, $a_k \in \mathrm{Hol}(V)$, $k = 0, 1, \dots$ and assume that for every $\widetilde{V} \Subset V$, $\exists C = C_{\widetilde{V}} > 0$ such that

$$|a_k(z)| \le C^{k+1} k^k, \; z \in \widetilde{V}. \tag{6}$$

$a = \sum_0^\infty a_k(z) h^k$ is called a formal classical analytic symbol.

We have a realization of a on $\widetilde{V}$:

$$a_{\widetilde{V}}(z; h) = \sum_{0 \le k \le (eC_{\widetilde{V}} h)^{-1}} a_k(z) h^k \in H_\Phi^{loc}(\widetilde{V}).$$

The following result was obtained in [RoSjVu18], and independently by Deleporte [De18], see also [HeLuXu17]. L. Charles [Ch19] and H. Hezari and H. Xu [HeXu19] have given other proofs.

Theorem

In the analytic case, the amplitude a in (1) is a realization of a classical analytic symbol. We have (5) with $\mathcal{O}(h^\infty)$ replaced by $\mathcal{O}\exp(-1/(Ch)$.

Our proof follows the main ideas of the one in [BeBeSj08], however, the procedure of repeated divisions turned out to be technically complicated for analytic symbols, so we used a variant, showing that we can go from an operator of the form (1) to (cf. (2))

$$Cu(x) = (2\pi h)^{-n} \iint e^{\frac{i}{h}(x-y)\cdot\eta} c\left(\frac{x+y}{2}, \eta; h\right) u(y)dyd\eta, \qquad (7)$$

by composing various integral transforms, and that the map $a \mapsto c$ is an elliptic analytic Fourier integral operator with a canonical transformation mapping the zero-section to the zero section. Such an operator conserves analytic symbols, and when elliptic, the same holds for the inverse. The proofs of [De18], [Ch19], [HeXu19] work more directly with analytic symbols, controling certain quasinorms appearing for these objects.

4. A direct argument without the Kuranishi trick

This has recently been developed in the analytic case with Hitrik and Deleporte [DeHiSj20].

Let Φ be st.pl.s.h. and real analytic. As before we try $\Pi : L^2_\Phi \to H_\Phi$ in the form:

$$\begin{aligned}\Pi u &= \iint e^{\frac{2}{h}\Psi(x,\overline{y})} a(x,\overline{y};h) e^{-\frac{2}{h}\Phi(y)} u(y) dy d\overline{y} \\ &= \iint e^{\frac{2}{h}\Psi(x,y^\dagger)} a(x,y^\dagger;h) e^{-\frac{2}{h}\Psi(y,y^\dagger)} u(y) dy dy^\dagger,\end{aligned}$$

where we think of $y^\dagger$ as an independent variable. We look for Π such that up to an exponentially small error,

$$(\Pi u|v)_\Phi = (u|v)_\Phi,\ \forall u,v \in H_\Phi. \tag{8}$$

At least formally this implies the reproducing property

$$\Pi u = u,\ \ u \in H_\Phi,$$

up to an exponentially small error.

Here

$$(u|v)_\Phi = \iint u(x)\overline{v}(x)e^{-\frac{2}{h}\Phi(x)}dxd\overline{x} = \iint u(x)v^\dagger(x^\dagger)e^{-\frac{2}{h}\Psi(x,x^\dagger)}dxdx^\dagger,$$

where $v^\dagger(x^\dagger) = \overline{v}(\overline{x^\dagger})$.
The left hand side in (8) is equal to

$$\iint \left(\iint e^{\frac{2}{h}\Psi(x,y^\dagger)-\frac{2}{h}\Psi(y,y^\dagger)}a(x,y^\dagger;h)u(y)dydy^\dagger\right)e^{-\frac{2}{h}\Psi(x,x^\dagger)}v^\dagger(x^\dagger)dxdx^\dagger$$

$$= \iint \left(\iint e^{\frac{2}{h}(\Psi(x,y^\dagger)-\Psi(y,y^\dagger)-\Psi(x,x^\dagger))}a(x,y^\dagger;h)dxdy^\dagger\right)u(y)v^\dagger(x^\dagger)dydx^\dagger,$$

so we get (8) if

$$\iint e^{\frac{2}{h}(\Psi(x,y^\dagger)-\Psi(y,y^\dagger)-\Psi(x,x^\dagger))}a(x,y^\dagger;h)dxdy^\dagger = e^{-\frac{2}{h}\Psi(y,x^\dagger)}), \qquad (9)$$

which we write as

$$\iint e^{\frac{2}{h}(\Psi(x,y^\dagger)-\Psi(y,y^\dagger)+\Psi(y,x^\dagger)-\Psi(x,x^\dagger))}a(x,y^\dagger;h)dxdy^\dagger = 1, \tag{10}$$

or

$$(Aa)(y,x^\dagger) = 1, \tag{11}$$

where A is a Fourier integral operator which takes functions of $(x,y^\dagger)$ to functions of $(y,x^\dagger)$. We check that A gives a bijection from the space of classical analytic symbols to itself and hence there is a unique symbol a in this class, that solves (11). □

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