Hyperbolic Problems with Totally Characteristic Boundary

Ingo Witt (University of Göttingen)

Joint with

Zhuoping Ruan (Nanjing University)

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We study the hyperbolic I(B)VP

$$(1) \qquad \begin{cases} \partial_t u + x\, A(t,x,y)\partial_x u + \sum_{j=1}^{d} A_j(t,x,y)\partial_j u = f(t,x,y), \\ u\big|_{t=0} = u_0(x,y). \end{cases}$$

where $(t,x,y) \in (0,T) \times \mathbb{R}_+ \times \mathbb{R}^d$, $\partial_j = \partial/\partial y_j$, and $A, A_j \in \mathscr{C}^\infty([0,T] \times \overline{\mathbb{R}}_+^{1+d}; M_{N\times N}(\mathbb{C}))$.

As we will see, no boundary conditions are to be imposed at $x = 0$ due to the fact that the boundary is totally characteristic.

We allow (formal) asymptotics of the form

$$u(t,x,y) \sim \sum_{(p,k)\in P} \frac{(-1)^k}{k!}\, x^{-p} \log^k x \, (\gamma_{pk} u)(t,y) \quad \text{as } x \to +0,$$

where $P \subset \mathbb{C} \times \mathbb{N}_0$ is an asymptotic type.

- We show well-posedness in function spaces $H^{s,\delta}_{P,\theta}(\overline{\mathbb{R}}^{1+d}_+)$.
- Results comparable to ours were obtained by R. Sakamoto (1989) (well-posedness in function spaces $H^{s,\delta}_{T^\delta P_0,s}(\overline{\mathbb{R}}^{1+d}_+)$ for $s \geq 0$, but with different techniques).
- New is that the boundary traces $\gamma_{pk} u$ solve hyperbolic Cauchy problems in the boundary $(0,T) \times \partial\overline{\mathbb{R}}^{1+d}_+$.

1. Hyperbolic problems with characteristic boundary of constant rank

2. Compressible Euler equations with vacuum boundary in 3D:

$$\begin{cases} \partial_t \rho + v \cdot \nabla \rho + \rho \operatorname{div} v = 0, \\ \partial_t v + v \cdot \nabla v + \dfrac{1}{\rho} \nabla p = 0, \\ \partial_t e + v \cdot \nabla e + \dfrac{p}{\rho} \operatorname{div} v = 0, \end{cases}$$

written in non-conservative coordinates $(\rho, v, e)^T$, where ρ is mass density, $v = (v_1, v_2, v_3)^T$ is velocity, e is specific internal energy, p is pressure, and c is speed of sound.

- Moving vacuum interface (Jang-Masmoudi, 2015): $c \sim \sqrt{x}$ as $x \to +0$, where $x > 0$ is the distance measured from inside the flow region.
- Stationary vacuum interface: $c \sim x$ as $x \to +0$. Here, we have, among others, that $\rho \sim x^{\frac{2}{\gamma-1}}$ as $x \to +0$, where $1 < \gamma < 3$ is the adiabatic constant.

We first need function spaces.

1. The weighted Sobolev space $\mathcal{H}^{s,\gamma}(\mathbb{R}^{1+d}_+)$ for $s \in \mathbb{N}_0$, $\gamma \in \mathbb{R}$ consists of all $u = u(x, y)$ such that

$$x^{-\gamma}(x\partial_x)^j \partial_y^\alpha u \in L^2(\mathbb{R}^{1+d}_+), \quad j + |\alpha| \leq s.$$

For general $s \in \mathbb{R}$, $\gamma \in \mathbb{R}$, these spaces are defined by interpolation and duality.

2. We also set

$$\mathcal{K}^{s,\gamma}(\mathbb{R}^{1+d}_+) = \left\{ u \mid \varphi u \in \mathcal{H}^{s,\gamma}(\mathbb{R}^{1+d}_+), \, (1-\varphi)\, u \in H^s(\mathbb{R}^{1+d}_+) \right\}.$$

Here, $\varphi \in \mathscr{C}^\infty(\overline{\mathbb{R}}_+)$, $\varphi(x) = 1$ for $x \leq 1$, and $\varphi(x) = 0$ for $x \geq 2$.

We now fix a basic conormal order $\delta \in \mathbb{R}$. The space $\mathcal{K}^{0,\delta}(\mathbb{R}^{1+d}_+)$ will be our basic Hilbert space.

An asymptotic type $P \in \underline{\mathrm{As}}^\delta$ is given by a discrete set $\pi_\mathbb{C} P \subset \mathbb{C}$ and sequence $\{m_p\}_{p \in \pi_\mathbb{C} P} \subset \mathbb{N}$ with the following properties:

- $\pi_\mathbb{C} P \subset \{z \in \mathbb{C} \mid \Re z < 1/2 - \delta\}$,
- $\Re p \to -\infty$ as $p \in \pi_\mathbb{C} P$, $|p| \to \infty$,
- $p \in \pi_\mathbb{C} P$ implies $p - 1 \in \pi_\mathbb{C} P$ and $m_{p-1} \geq m_p$.

It is convenient to write P as a set, i.e.,
$P = \{(p, k) \in \mathbb{C} \times \mathbb{N}_0 \mid p \in \pi_\mathbb{C} P,\ k < m_p\}$.

Due to lack of regularity, we cannot write asymptotic terms as tensor products. Instead, for $w \in H^{s,\langle k\rangle}(\mathbb{R}^d)$, we have

$$\mathcal{F}^{-1}\{\varphi(x\langle\eta\rangle)\hat{w}(\eta)\}x^{-p}\log^k x \in \bigcap_{\epsilon>0} \mathcal{H}^{s+\epsilon,1/2-\Re p-\epsilon}(\mathbb{R}^{1+d}_+)$$

and

$$\mathcal{F}^{-1}\{\varphi(x\langle\eta\rangle)\hat{w}(\eta)\}x^{-p}\log^k x - \varphi(x)x^{-p}\log^k x\, w(y) \in \bigcap_{\epsilon>0} \mathcal{H}^{s-\epsilon,1/2-\Re p+\epsilon}(\mathbb{R}^{1+d}_+)$$

$w \in H^{s,\langle k\rangle}(\mathbb{R}^d)$ means that $\langle\eta\rangle^s \log^k\langle\eta\rangle\hat{w}(\eta) \in L^2(\mathbb{R}^d)$. Here, $\langle\eta\rangle = (2+|\eta|^2)^{1/2}$.

For $(p,k) \in P$, define the potential operator Γ_{pk} by

$$(\Gamma_{pk} w)(x,y) = \frac{(-1)^k}{k!} \mathcal{F}^{-1} \{\varphi(x\langle\eta\rangle)\hat{w}(\eta)\} \, x^{-p} \log^k x.$$

Definition

Let $s \in \mathbb{R}$, $P \in \underline{\mathrm{As}}^{\delta}$, $\theta \geq 0$. For $\pi_{\mathbb{C}} P \cap \{z \in \mathbb{C} \mid \Re z = 1/2 - \delta - \theta\} = \emptyset$, the space $H^{s,\delta}_{P,\theta}(\overline{\mathbb{R}}^{1+d}_+)$ consists of all $u \in \mathcal{K}^{s,\delta}(\mathbb{R}^{1+d}_+)$ for which there are

$$u_{pk} \in H^{s+\Re p+\delta-1/2,\langle k\rangle}(\mathbb{R}^d)$$

for $(p,k) \in P$, $\Re p > 1/2 - \delta - \theta$ such that

$$\varphi(x)u(x,y) - \sum_{\substack{(p,k)\in P,\\ \Re p > 1/2-\delta-\theta}} (\Gamma_{pk} u_{pk})(x,y) \in \mathcal{H}^{s-\theta,\delta+\theta}(\mathbb{R}^{1+d}_+).$$

For general $\theta \geq 0$, these spaces are defined by complex interpolation.

We shall write $\gamma_{pk} u = u_{pk}$.

Lemma

For $s \geq 0$,

- $H^s(\mathbb{R}^{1+d}_+) = H^{s,0}_{P_0,s}(\overline{\mathbb{R}}^{1+d}_+)$, *where $P_0 = \{(-\ell, 0) \mid \ell \in \mathbb{N}_0\}$ is the asymptotic type that arises from Taylor expansion,*
- $H^s_0(\overline{\mathbb{R}}^{1+d}_+) = H^{s,0}_{\mathcal{O},s}(\overline{\mathbb{R}}^{1+d}_+)$, *where $\mathcal{O}$ is the empty asymptotic type.*

Let $\Psi_c^\mu(\overline{\mathbb{R}}_+^{1+d})$ denote the class of classical cone-degenerate pseudodifferential operators on $\mathbb{R}_+^{1+d}$, of order $\mu \in \mathbb{R}$, with holomorphic conormal symbols.

We formally write $A(x, y, xD_x, D_y)$ for elements of $\Psi_c^\mu(\overline{\mathbb{R}}_+^{1+d})$. Then $\sigma_c^{-j}(A)(z) = \frac{1}{j!}\,\partial_x^j A(0, y, \mathrm{i}z, D_y) \in \Psi^\mu(\mathbb{R}^d)$ for $j \in \mathbb{N}_0$.

Elements of the calculus

- $\Psi_c^\mu(\overline{\mathbb{R}}_+^{1+d}) \circ \Psi_c^\nu(\overline{\mathbb{R}}_+^{1+d}) \subseteq \Psi_c^{\mu+\nu}(\overline{\mathbb{R}}_+^{1+d})$ plus symbolic rules for the composition.
- Formal adjoints belong to the calculus as well.
- $\Psi_c^\mu(\overline{\mathbb{R}}_+^{1+d}) \subset \bigcap_{s,\delta,P,\theta} \mathcal{L}(H_{P,\theta}^{s+\mu,\delta}(\overline{\mathbb{R}}_+^{1+d}), H_{P,\theta}^{s,\delta}(\overline{\mathbb{R}}_+^{1+d}))$

Important note

Symmetrization of a hyperbolic systems requires neither inversion of nor a parametrix construction for elliptic operators.

We now study the Cauchy problem

$$(2) \qquad \begin{cases} \partial_t u + \mathcal{A}(t, x, y, xD_x, D_y)u = f(t, x, y) & \text{in } (0, T) \times \mathbb{R}^{1+d}_+, \\ u\big|_{t=0} = u_0(x, y) & \text{on } \mathbb{R}^{1+d}_+, \end{cases}$$

where $\mathcal{A} \in \mathscr{C}^\infty([0, T]; \Psi^1_{\mathrm{c}}(\overline{\mathbb{R}}^{1+d}_+; \mathbb{C}^N))$.

Hyperbolicity assumption (existence of a symbolic symmetrizer)

There exists a $b \in \mathscr{C}^\infty([0, T]; S^{(0)}(\widetilde{T}^*\overline{\mathbb{R}}^{1+d}_+ \setminus 0; \mathrm{Mat}_{N\times N}(\mathbb{C})))$ such that

- $b(t)$ is positive definite,
- $b(t)\,\widetilde{\sigma}^1(\mathcal{A}(t))$ is skew-Hermitian.

Theorem (Existence, uniqueness, and higher regularity)

Let $u_0 \in H^{s,\delta}_{P,\theta}(\overline{\mathbb{R}}^{1+d}_+;\mathbb{C}^N)$ *and* $f \in \bigcap_{k=0}^{\ell} W^{k,1}((0,T);H^{s-k,\delta}_{P,\theta_k}(\overline{\mathbb{R}}^{1+d}_+;\mathbb{C}^N))$ *for some* $\ell \in \mathbb{N}_0$*, where* $\theta = \theta_0 \geq \theta_1 \geq \ldots \geq \theta_\ell$*. Then the Cauchy problem* (2) *possesses a unique solution*

$$u \in \bigcap_{k=0}^{\ell} \mathscr{C}^k([0,T];H^{s-k,\delta}_{P,\theta_k}(\overline{\mathbb{R}}^{1+d}_+;\mathbb{C}^N)).$$

Furthermore, the boundary trace $\gamma_{pk}u$ *for* $(p,k) \in P$*,* $\Re p > 1/2 - \delta - \theta$ *solves the problem*

(3)
$$\begin{cases} \partial_t(\gamma_{pk}u) + \sigma_c^0(\mathcal{A}(t))(p)\gamma_{pk}u = \gamma_{pk}f \\ \qquad - \sum\limits_{\substack{j\geq 0,\, l-r=k, \\ (j,l,r)\neq(0,k,0)}} \dfrac{1}{r!}\,\partial_z^r\sigma_c^{-j}(\mathcal{A}(t))(p+j)\,\gamma_{p+j,l}(u), \\ \gamma_{pk}u\big|_{t=0} = \gamma_{pk}u_0. \end{cases}$$

This is Cauchy problem for an $N \times N$ first-order hyperbolic system in $(0,T) \times \mathbb{R}^d$.

Hyperbolicity is implied by $\sigma^1(\sigma_c^0(\mathcal{A}(t))(p))(y,\eta) = \tilde{\sigma}^1(\mathcal{A}(t))(0,y,0,\eta)$.

The main step in the proof is to establish a corresponding *a priori* estimate.

Theorem

Let $u \in \mathscr{C}([0,T]; H^{s+1,\delta}_{P,\theta}(\overline{\mathbb{R}}^{1+d}_+; \mathbb{C}^N)) \cap \mathscr{C}^1([0,T]; H^{s,\delta}_{P,\theta}(\overline{\mathbb{R}}^{1+d}_+; \mathbb{C}^N))$*. Then*

$$\max_{0\le t\le T} \|u(t)\|_{H^{s,\delta}_{P,\theta}} \lesssim \|u(0)\|_{H^{s,\delta}_{P,\theta}} + \int_0^T \|\partial_t u(t) + \mathcal{A}(t)u(t)\|_{H^{s,\delta}_{P,\theta}}\, dt.$$

Utilizing an order reduction, we can first reduce to the case $s = 0$.

Then we split the proof into two parts:

- $\theta = 0$ and $H^{0,\delta}_{P,0}(\overline{\mathbb{R}}^{1+d}_+) = \mathcal{K}^{0,\delta}(\mathbb{R}^{1+\delta}_+)$ (i.e., without asymptotics),
- $\theta > 0$ (i.e., possibly with asymptotics).

Let $\mathcal{B} \in \mathscr{C}^\infty([0, T]; \Psi_c^0(\overline{\mathbb{R}}_+^{1+d}; \mathbb{C}^N))$ be a symmetrizer for $\partial_t + \mathcal{A}(t)$, i.e.,

- $\mathcal{B}(t) = \mathcal{B}(t)^* \geq c\,\mathrm{I}$ for some $c > 0$,
- $\Im\,\widetilde{\sigma}^1(\mathcal{B}\mathcal{A}) = 0$.

In fact, we can choose $\mathcal{B}$ such that $\widetilde{\sigma}^0(\mathcal{B}(t)) = b(t)$.

Derivation of the energy inequality in case $\theta = 0$ relies on the following facts:

- $\langle \mathcal{B}(t)v, v\rangle$ is equivalent to $\|v\|^2$ uniformly in $t \in [0, T]$.
- Integration by parts produces no boundary terms, i.e., $\langle Au, v\rangle = \langle u, A^*v\rangle$ for $A \in \Psi_c^1(\overline{\mathbb{R}}_+^{1+d})$ and $u, v \in \mathcal{K}^{1,\delta}(\mathbb{R}_+^{1+d})$.
- $\mathcal{B}\mathcal{A} + (\mathcal{B}\mathcal{A})^* \in \mathscr{C}^\infty([0, T]; \Psi_c^0(\overline{\mathbb{R}}_+^{1+d}; \mathbb{C}^N))$.

In case $\theta > 0$, we can assume $\pi_{\mathbb{C}} P \cap \{z \in \mathbb{C} \mid \Re z = 1/2 - \delta - \theta\} = \emptyset$.

Write $u_0 = u(0)$ and $f = \partial_t u + \mathcal{A} u$.

Successively solving the Cauchy problems (3) for the boundary traces $\gamma_{pk} u$, we find

$$\max_{0 \le t \le T} \|\gamma_{pk} u(t)\|_{H^{s+\Re p+\delta-1/2,\langle k\rangle}} \lesssim$$
$$\sum_{j\ge 0,\, l \ge k} \left(\|\gamma_{p+j,l} u_0\|_{H^{s+\Re p+\delta+j-1/2,\langle l\rangle}} + \int_0^T \|\gamma_{p+j,l} f(t)\|_{H^{s+\Re p+\delta+j-1/2,\langle l\rangle}}\, dt \right).$$

Setting $\overline{u}_0 = u_0 - \sum_{\substack{(p,k)\in P,\\ \Re p > 1/2-\delta-\theta}} \left(\Gamma_{pk}\gamma_{pk} u\right)\big|_{t=0} \in \mathcal{K}^{s-\theta+1,\delta+\theta}(\mathbb{R}_+^{1+\delta};\mathbb{C}^N)$,
$\overline{f} = f - (\partial_t + \mathcal{A}(t))\left(\sum_{\substack{(p,k)\in P,\\ \Re p > 1/2-\delta-\theta}} \Gamma_{pk}\gamma_{pk} u\right) \in$
$\mathscr{C}([0,T];\mathcal{K}^{s-\theta,\delta+\theta}(\mathbb{R}_+^{1+\delta};\mathbb{C}^N))$, we next solve the hyperbolic system

$$\begin{cases} \partial_t \overline{u} + \mathcal{A}(t)\overline{u} = \overline{f}, \\ \overline{u}\big|_{t=0} = \overline{u}_0. \end{cases}$$

Then

$$\max_{0\le t\le T} \|\overline{u}(t)\|_{\mathcal{K}^{s-\theta,\delta+\theta}} \lesssim \|\overline{u}_0\|_{\mathcal{K}^{s-\theta,\delta+\theta}} + \int_0^T \|\overline{f}(t)\|_{\mathcal{K}^{s-\theta,\delta+\theta}}\, dt.$$

Because of $u = \overline{u} + \sum_{\substack{(p,k)\in P, \\ \Re p > 1/2-\delta-\theta}} \Gamma_{pk}\gamma_{pk} u$, it follows that

$$\max_{0\le t\le T} \|u(t)\|_{H^{s,\delta}_{P,\theta}} \lesssim \|u_0\|_{H^{s,\delta}_{P,\theta}} + \int_0^T \|f(t)\|_{H^{s,\delta}_{P,\theta}}\, dt.$$

□