Liftings for ultra-modulation spaces, and one-parameter groups of Gevrey type pseudo-differential operators

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Plan of the talk

1. Motivations and preliminaries
2. More general lifting results

Papers in background

- P. Boggiatto, E. Cordero, K. Gröchenig *Generalized Anti-Wick Operators with Symbols in Distributional Sobolev spaces*, Research Report, Quaderni Dip. Mat. Univ. Torino, **32**, 2002.
- P. Boggiatto, J. Toft *Embeddings and compactness for generalized SobolevShubin spaces and modulation spaces*, Applicable Analysis **84** (2005), 269–282.
- K. Gröchenig, J. Toft *Isomorphism properties of Toeplitz operators and pseudo-differential operators between modulation spaces*, J. Anal. Math. **114** (2011), 255–283.
- K. Gröchenig, J. Toft *The range of localization operators and lifting theorems for modulation and Bargmann-Fock spaces*, Trans. Amer. Math. Soc. **365** (2013), 4475–4496.
- A. Abdeljawad, S. Coriasco, J. Toft *Liftings for ultra-modulation spaces, and one-parameter groups of Gevrey type pseudo-differential operators*, Anal. Appl. **18** (2020), 523–583.

What is lifting

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The spaces V_1 and V_2 are said to perform lifting property if there is a "convenient" map T which maps V_1 bijectively into V_2. The map T is then said to lift V_1 into V_2.

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Example: Let $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$ and

$$L^p_s(\mathbf{R}^d) = \{\, f\in L^p_{loc}(\mathbf{R}^d)\, ;\, \|f\|_{L^p_s} \equiv \|f\cdot \langle\,\cdot\,\rangle^s\|_{L^p} < \infty \,\}.$$

Then $f\mapsto f\cdot\langle\,\cdot\,\rangle^{s_0}$ lifts L^p_s to $L^p_{s-s_0}$.

Example: The Sobolev space:

$$H^p_\sigma(\mathbf{R}^d) = \{\, f\in \mathscr{S}'(\mathbf{R}^d)\, ;\, \|f\|_{H^p_\sigma} \equiv \|\langle D\rangle^\sigma f\|_{L^p} < \infty \,\}$$

Then $f\mapsto \langle D\rangle^{\sigma_0} f$ lifts H^p_σ to $H^p_{\sigma-\sigma_0}$

Example: The Sobolev Kato space:

$$H^2_{s,\sigma}(\mathbf{R}^d) = \{\, f\in \mathscr{S}'(\mathbf{R}^d)\, ;\, \|f\|_{H^2_{s,\sigma}} \equiv \|\langle x\rangle^s\langle D\rangle^\sigma f\|_{L^2} < \infty \,\}$$

Then $f\mapsto \langle x\rangle^{s_0}\langle D\rangle^{\sigma_0} f$ lifts $H^2_{s,\sigma}$ to $H^2_{s-s_0,\sigma-\sigma_0}$.

Preparations for more liftings - ΨDO and Toeplitz Ops.

Let $a \in \mathscr{S}'(\mathbf{R}^{2d})$ and let $\mathscr{F}$ be the Fourier transform given by

$$(\mathscr{F}f)(\xi) = \widehat{f}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} f(y) e^{-i\langle y,\xi\rangle}\, dy$$

when $f \in L^1(\mathbf{R}^d)$.

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- Let $\phi \in \mathscr{S}(\mathbf{R}^d)$ and $f \in \mathscr{S}'(\mathbf{R}^d)$. Then

$$V_\phi f(x,\xi) = \mathscr{F}(f \cdot \overline{\phi(\,\cdot\, - x)})(\xi)$$

is the Short-Time Fourier Transform (STFT) of f w.r.t. ϕ.

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We have $\quad \operatorname{Tp}(a) = \operatorname{Op}^w(a * W_{\phi,\phi})$, where

$$W_{\phi_1,\phi_2}(x,\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} \phi_1(x-y/2)\overline{\phi_2(x+y/2)}e^{i\langle y,\xi\rangle}\,dy$$

is the Wigner distribution.

Preparations for other liftings - Modulation spaces

- A moderate weight ω on $\mathbf{R}^d$ is a positive function on $\mathbf{R}^d$ such that for some positive function v we have

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- We let $\mathscr{P}(\mathbf{R}^d)$ be all $\omega \in \mathscr{P}_E(\mathbf{R}^d)$ such that for some $N \geq 0$,

$$\omega(x+y) \lesssim \omega(x)\langle y\rangle^N, \qquad x,y \in \mathbf{R}^d. \tag{*}''$$

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- The Gelfand-Shilov space $\Sigma_1(\mathbf{R}^d)$ ($\mathcal{S}_1(\mathbf{R}^d)$) consists of all $f \in L^\infty(\mathbf{R}^d) \cap \mathscr{F}L^\infty(\mathbf{R}^d)$ such that

  $$|f(x)| \lesssim e^{-r|x|} \quad \text{and} \quad |\widehat{f}(\xi)| \lesssim e^{-r|\xi|}$$

  for every $r > 0$ (for some $r > 0$).

  Equivalently, $f \in \Sigma_1(\mathbf{R}^d)$ ($f \in \mathcal{S}_1(\mathbf{R}^d)$), iff

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- We have $\Sigma_1 \subseteq \mathcal{S}_1 \subseteq \mathscr{S}$ and $\mathscr{S}' \subseteq \mathcal{S}_1' \subseteq \Sigma_1'$.

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(Feichtinger 1983) Let $p, q \in (0, \infty]$, $\omega \in \mathscr{P}_E(\mathbf{R}^{2d})$, and $\phi \in \Sigma_1(\mathbf{R}^d) \setminus 0$. Then the modulation space $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ consists of all $f \in \Sigma_1'(\mathbf{R}^d)$ such that

$$\|f\|_{M^{p,q}_{(\omega)}} \equiv \left(\int_{\mathbf{R}^d} (|V_\phi f(x,\xi)\omega(x,\xi)|^p)^{q/p} \, d\xi \right)^{1/q} < \infty.$$

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$M^{p,q}_{(\omega)}(\mathbf{R}^d)$ is independent of ϕ.

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We have:

$$\mathcal{S}_1(\mathbf{R}^d)\subseteq M^{p,q}_{(\omega)}(\mathbf{R}^d)\subseteq \mathcal{S}_1'(\mathbf{R}^d) \quad \Leftrightarrow \quad \omega\in\mathscr{P}^0_E(\mathbf{R}^{2d}).$$

and

$$\mathscr{S}(\mathbf{R}^d)\subseteq M^{p,q}_{(\omega)}(\mathbf{R}^d)\subseteq \mathscr{S}'(\mathbf{R}^d) \quad \Leftrightarrow \quad \omega\in\mathscr{P}(\mathbf{R}^{2d}).$$

Some properties

Let $\langle x\rangle = (1+|x|^2)^{1/2}$, and

$$M^{p,q}_{s,t} = M^{p,q}_{(\omega)} \quad \text{when} \quad \omega(x,\xi) = \langle x\rangle^t \langle \xi\rangle^s$$

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- $M^{2,2}_{s,0} = H^2_s, \quad M^{2,2}_{0,s} = L^2_s$
- if $\omega_0(x,\xi) = \omega(-\xi,x)$, then $\|\mathscr{F} f\|_{M^{p,p}_{(\omega_0)}} \asymp \|f\|_{M^{p,p}_{(\omega)}}$
- if $1 \le p,q < \infty$, then $(M^{p,q}_{(\omega)})' = M^{p',q'}_{(1/\omega)} \qquad (1/p + 1/p' = 1)$
- if $p_1 \le p_2$, $q_1 \le q_2$ and $\omega_2 \lesssim \omega_1$ then $M^{p_1,q_1}_{(\omega_1)} \subseteq M^{p_2,q_2}_{(\omega_2)}$
- Modulation spaces posses convenient discretization properties using Gabor frames

For future references:

- if $\omega \in \mathscr{P}$, then $S^{(\omega)} = \{\, a \in C^\infty \, ; \, \partial^\alpha a \lesssim \omega \,\}$.
- $S^0_{0,0} = S^{(1)}$.
- if $\omega(X+Y) \lesssim \omega(X) e^{r|Y|^{\frac{1}{s}}}$ for every $r > 0$, then let $\Gamma^{(\omega)}_s = \{\, a \in C^\infty \, ; \, \partial^\alpha a \lesssim h^{|\alpha|} \alpha!^s \omega$ for some $h > 0 \,\}$.
- By Cordero-Nicola-Rodino: $\Gamma^{(\omega_1)}_s \# \Gamma^{(\omega_2)}_s \subseteq \Gamma^{(\omega_1 \omega_2)}_s$
- Let $\Gamma^0_{s;0} = \Gamma^{(1)}_s$. Then $\Gamma^0_{s;0} \subseteq S^0_{0,0}$.

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Then $M^2_{(\omega_s)}(\mathbf{R}^d) = Q_s(\mathbf{R}^d)$, and $\operatorname{Tp}(\omega_s)$ lifts $M^2_{(\omega_s)}(\mathbf{R}^d)$ to $M^2(\mathbf{R}^d) = L^2(\mathbf{R}^d)$.

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  Notice that in contrast to earlier lifting results, the weights and operator symbols are not of split form - i.e. not of the form $\omega_1(x)\omega_2(\xi)$. This make it more complicated because of strong interactions between differentiations and space variables (multiplications).

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- Gröchenig-T. 2013: Let $\omega_0, \omega \in \mathscr{P}^0_E(\mathbf{R}^{2d})$ be radial symmetric in each phase-shift variable, i.e.
$$\omega(x,\xi) = \widetilde{\omega}(r_1, \dots, r_d), \quad r_j = (x_j^2 + \xi_j^2)^{\frac{1}{2}}.$$
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3. Abdeljawad-Coriasco-T. 2017/2020: Let $\omega_0, \omega \in \mathscr{P}^0_E(\mathbf{R}^{2d})$. Then $\mathrm{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

Liftings again

1. Gröchenig-T. 2011: Let $\omega_0, \omega \in \mathscr{P}(\mathbf{R}^{2d})$. Then $\mathrm{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.
2. Gröchenig-T. 2013: Let ϕ be a Gaussian, and $\omega_0, \omega \in \mathscr{P}^0_E(\mathbf{R}^{2d})$ be radial in each phase-shift variable, i.e.

$$\omega(x,\xi) = \widetilde{\omega}(r_1,\dots,r_d), \quad r_j = (x_j^2+\xi_j^2)^{\frac{1}{2}}.$$

   Then $\mathrm{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.
3. Abdeljawad-Coriasco-T. 2017/2020: Let $\omega_0, \omega \in \mathscr{P}^0_E(\mathbf{R}^{2d})$. Then $\mathrm{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

Liftings again

1. Gröchenig-T. 2011: Let $\omega_0, \omega \in \mathscr{P}(\mathbf{R}^{2d})$. Then $\mathrm{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.
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$$\omega(x,\xi) = \widetilde{\omega}(r_1, \dots, r_d), \quad r_j = (x_j^2 + \xi_j^2)^{\frac{1}{2}}.$$

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The approach to prove (3) is similar to (1), and we now explain details.

Arguments in the proof

Gröchenig-T. 2011

Let $\omega_0, \omega \in \mathscr{P}(\mathbf{R}^{2d})$. Then $\mathrm{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

Arguments in the proof

Gröchenig-T. 2011

Let $\omega_0, \omega \in \mathscr{P}(\mathbf{R}^{2d})$. Then $\operatorname{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

(1) $\operatorname{Tp}_\phi(\omega_0) = \operatorname{Op}^w(\widetilde{\omega}_0)$, where $\widetilde{\omega}_0 = \omega_0 * W_{\phi,\phi} \in S^{(\omega_0)}$ and $\widetilde{\omega}_0 \asymp \omega_0$. This gives $\operatorname{Tp}_\phi(\omega_0)$ is continuous from $M^{p,q}_{(\omega)}$ to $M^{p,q}_{(\omega/\omega_0)}$.

Arguments in the proof

Gröchenig-T. 2011

Let $\omega_0, \omega \in \mathscr{P}(\mathbf{R}^{2d})$. Then $\operatorname{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

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This gives $\operatorname{Tp}_\phi(\omega_0)$ is continuous from $M^{p,q}_{(\omega)}$ to $M^{p,q}_{(\omega/\omega_0)}$.

(2) $\vartheta = \omega_0^{\frac{1}{2}}$. Then $\operatorname{Tp}(\omega_0)$ lifts $M^2_{(\vartheta)}$ to $M^2_{(1/\vartheta)}$ Let T_1 be the inverse.

Arguments in the proof

Gröchenig-T. 2011

Let $\omega_0, \omega \in \mathscr{P}(\mathbf{R}^{2d})$. Then $\operatorname{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

(1) $\operatorname{Tp}_\phi(\omega_0) = \operatorname{Op}^w(\widetilde{\omega}_0)$, where $\widetilde{\omega}_0 = \omega_0 * W_{\phi,\phi} \in S^{(\omega_0)}$ and $\widetilde{\omega}_0 \asymp \omega_0$.
This gives $\operatorname{Tp}_\phi(\omega_0)$ is continuous from $M^{p,q}_{(\omega)}$ to $M^{p,q}_{(\omega/\omega_0)}$.

(2) $\vartheta = \omega_0^{\frac{1}{2}}$. Then $\operatorname{Tp}(\omega_0)$ lifts $M^2_{(\vartheta)}$ to $M^2_{(1/\vartheta)}$ Let T_1 be the inverse.

$$\|f\|_{M^2_{(\vartheta)}} = \sup_{\|g\|_{M^2_{(\vartheta)}} \le 1} |(f,g)_{M^2_{(\vartheta)}}| \asymp \sup_{\|g\|_{M^2_{(\vartheta)}} \le 1} \left| \int V_\phi f(x,\xi)\, \overline{V_\phi f(x,\xi)} \vartheta(x,\xi)^2 \, dx d\xi \right|$$

$$= \sup_{\|g\|_{M^2_{(\vartheta)}} \le 1} |(\omega_0 \cdot V_\phi f, V_\phi g)_{L^2(\mathbf{R}^{2d})}| = \sup_{\|g\|_{M^2_{(\vartheta)}} \le 1} |(\operatorname{Tp}_\phi(\omega_0) f, g)_{L^2(\mathbf{R}^d)}|$$

$$\asymp \| \operatorname{Tp}_\phi(\omega_0) f\|_{M^2_{(1/\vartheta)}}.$$

Arguments in the proof

Gröchenig-T. 2011

Let $\omega_0, \omega \in \mathscr{P}(\mathbf{R}^{2d})$. Then $\operatorname{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

(1) $\operatorname{Tp}_\phi(\omega_0) = \operatorname{Op}^w(\widetilde{\omega}_0)$, where $\widetilde{\omega}_0 = \omega_0 * W_{\phi,\phi} \in S^{(\omega_0)}$ and $\widetilde{\omega}_0 \asymp \omega_0$.
This gives $\operatorname{Tp}_\phi(\omega_0)$ is continuous from $M^{p,q}_{(\omega)}$ to $M^{p,q}_{(\omega/\omega_0)}$.

(2) $\vartheta = \omega_0^{\frac{1}{2}}$. Then $\operatorname{Tp}(\omega_0)$ lifts $M^2_{(\vartheta)}$ to $M^2_{(1/\vartheta)}$. Let T_1 be the inverse.

Arguments in the proof

Gröchenig-T. 2011

Let $\omega_0, \omega \in \mathscr{P}(\mathbf{R}^{2d})$. Then $\operatorname{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

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This gives $\operatorname{Tp}_\phi(\omega_0)$ is continuous from $M^{p,q}_{(\omega)}$ to $M^{p,q}_{(\omega/\omega_0)}$.

(2) $\vartheta = \omega_0^{\frac{1}{2}}$. Then $\operatorname{Tp}(\omega_0)$ lifts $M^2_{(\vartheta)}$ to $M^2_{(1/\vartheta)}$. Let T_1 be the inverse.

(3) By Bony-Chemin: There are $a \in S^{(\vartheta)}$ and $b \in S^{(1/\vartheta)}$ such that $\operatorname{Op}^w(a) \circ \operatorname{Op}^w(b) = \operatorname{Op}^w(a) \circ \operatorname{Op}^w(b) = \operatorname{Id}_{\mathscr{S}'}$.

Arguments in the proof

Gröchenig-T. 2011

Let $\omega_0, \omega \in \mathscr{P}(\mathbf{R}^{2d})$. Then $\mathrm{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

(1) $\mathrm{Tp}_\phi(\omega_0) = \mathrm{Op}^w(\widetilde{\omega}_0)$, where $\widetilde{\omega}_0 = \omega_0 * W_{\phi,\phi} \in S^{(\omega_0)}$ and $\widetilde{\omega}_0 \asymp \omega_0$.
This gives $\mathrm{Tp}_\phi(\omega_0)$ is continuous from $M^{p,q}_{(\omega)}$ to $M^{p,q}_{(\omega/\omega_0)}$.

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(4) By (2)–(3): $\mathrm{Op}^w(c) \equiv \mathrm{Op}^w(b) \circ \mathrm{Tp}(\omega_0) \circ \mathrm{Op}^w(b)$ is bijective on L^2.

Arguments in the proof

Gröchenig-T. 2011

Let $\omega_0, \omega \in \mathscr{P}(\mathbf{R}^{2d})$. Then $\mathrm{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

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This gives $\mathrm{Tp}_\phi(\omega_0)$ is continuous from $M^{p,q}_{(\omega)}$ to $M^{p,q}_{(\omega/\omega_0)}$.

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By symbolic calculus: $c \in S^{(1)} = S^0_{0,0}$.

Arguments in the proof

Gröchenig-T. 2011

Let $\omega_0, \omega \in \mathscr{P}(\mathbf{R}^{2d})$. Then $\operatorname{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

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This gives $\operatorname{Tp}_\phi(\omega_0)$ is continuous from $M^{p,q}_{(\omega)}$ to $M^{p,q}_{(\omega/\omega_0)}$.

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(4) $\operatorname{Op}^w(c) \equiv \operatorname{Op}^w(b) \circ \operatorname{Tp}(\omega_0) \circ \operatorname{Op}^w(b)$ is bijective on L^2. Symb. calc.: $c \in S^0_{0,0}$.

(5) Since $S^0_{0,0}$ is a Wiener algebra, c_1 in $\operatorname{Op}^w(c_1) \equiv \operatorname{Op}^w(c)^{-1}$ belongs to $S^0_{0,0}$. This gives $T_1 = \operatorname{Op}^w(b) \circ \operatorname{Op}^w(c_1) \circ \operatorname{Op}^w(b) \in \operatorname{Op}^w(S^{(1/\omega_0)})$.

Arguments in the proof

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(2) $\vartheta = \omega_0^{\frac{1}{2}}$. Then $\operatorname{Tp}(\omega_0)$ lifts $M^2_{(\vartheta)}$ to $M^2_{(1/\vartheta)}$. Let T_1 be the inverse.

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From these properties: By (1), (5) and general continuity results:

$$\operatorname{Tp}(\omega_0) \,:\, M^{p,q}_{(\omega)} \to M^{p,q}_{(\omega/\omega_0)}, \quad T_1 \,:\, M^{p,q}_{(\omega/\omega_0)} \to M^{p,q}_{(\omega)}$$

are continuous.

Arguments in the proof

Gröchenig-T. 2011

Let $\omega_0, \omega \in \mathscr{P}(\mathbf{R}^{2d})$. Then $\mathrm{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

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(2) $\vartheta = \omega_0^{\frac{1}{2}}$. Then $\mathrm{Tp}(\omega_0)$ lifts $M^2_{(\vartheta)}$ to $M^2_{(1/\vartheta)}$. Let T_1 be the inverse.

(3) By Bony-Chemin: There are $a \in S^{(\vartheta)}$ and $b \in S^{(1/\vartheta)}$ such that $\mathrm{Op}^w(a) \circ \mathrm{Op}^w(b) = \mathrm{Op}^w(a) \circ \mathrm{Op}^w(b) = \mathrm{Id}_{\mathscr{S}'}$.

(4) $\mathrm{Op}^w(c) \equiv \mathrm{Op}^w(b) \circ \mathrm{Tp}(\omega_0) \circ \mathrm{Op}^w(b)$ is bijective on L^2. Symb. calc.: $c \in S^0_{0,0}$.

(5) Since $S^0_{0,0}$ is a Wiener algebra, c_1 in $\mathrm{Op}^w(c_1) \equiv \mathrm{Op}^w(c)^{-1}$ belongs to $S^0_{0,0}$. This gives $T_1 = \mathrm{Op}^w(b) \circ \mathrm{Op}^w(c_1) \circ \mathrm{Op}^w(b) \in \mathrm{Op}^w(S^{(1/\omega_0)})$.

From these properties: By (1), (5) and general continuity results:

$$\mathrm{Tp}(\omega_0) : M^{p,q}_{(\omega)} \to M^{p,q}_{(\omega/\omega_0)}, \quad T_1 : M^{p,q}_{(\omega/\omega_0)} \to M^{p,q}_{(\omega)}$$

are continuous. By (2), $\mathrm{Tp}(\omega_0) \circ T_1 = T_1 \circ \mathrm{Tp}(\omega_0) = \mathrm{Id}_{\mathscr{S}}$.

Arguments in the proof

Gröchenig-T. 2011

Let $\omega_0, \omega \in \mathscr{P}(\mathbf{R}^{2d})$. Then $\operatorname{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

(1) $\operatorname{Tp}_\phi(\omega_0) = \operatorname{Op}^w(\widetilde{\omega}_0)$, where $\widetilde{\omega}_0 = \omega_0 * W_{\phi,\phi} \in S^{(\omega_0)}$ and $\widetilde{\omega}_0 \asymp \omega_0$.
This gives $\operatorname{Tp}_\phi(\omega_0)$ is continuous from $M^{p,q}_{(\omega)}$ to $M^{p,q}_{(\omega/\omega_0)}$.

(2) $\vartheta = \omega_0^{\frac{1}{2}}$. Then $\operatorname{Tp}(\omega_0)$ lifts $M^2_{(\vartheta)}$ to $M^2_{(1/\vartheta)}$. Let T_1 be the inverse.

(3) By Bony-Chemin: There are $a \in S^{(\vartheta)}$ and $b \in S^{(1/\vartheta)}$ such that
$\operatorname{Op}^w(a) \circ \operatorname{Op}^w(b) = \operatorname{Op}^w(a) \circ \operatorname{Op}^w(b) = \operatorname{Id}_{\mathscr{S}'}$.

(4) $\operatorname{Op}^w(c) \equiv \operatorname{Op}^w(b) \circ \operatorname{Tp}(\omega_0) \circ \operatorname{Op}^w(b)$ is bijective on L^2. Symb. calc.: $c \in S^0_{0,0}$.

(5) Since $S^0_{0,0}$ is a Wiener algebra, c_1 in $\operatorname{Op}^w(c_1) \equiv \operatorname{Op}^w(c)^{-1}$ belongs to
$S^0_{0,0}$. This gives $T_1 = \operatorname{Op}^w(b) \circ \operatorname{Op}^w(c_1) \circ \operatorname{Op}^w(b) \in \operatorname{Op}^w(S^{(1/\omega_0)})$.

From these properties: By (1), (5) and general continuity results:

$$\operatorname{Tp}(\omega_0) \,:\, M^{p,q}_{(\omega)} \to M^{p,q}_{(\omega/\omega_0)}, \quad T_1 \,:\, M^{p,q}_{(\omega/\omega_0)} \to M^{p,q}_{(\omega)}$$

are continuous. By (2), $\operatorname{Tp}(\omega_0) \circ T_1 = T_1 \circ \operatorname{Tp}(\omega_0) = \operatorname{Id}_{\mathscr{S}}$. By duality,
$\operatorname{Tp}(\omega_0) \circ T_1 = T_1 \circ \operatorname{Tp}(\omega_0) = \operatorname{Id}_{\mathscr{S}'}$.

Arguments in the proof

Gröchenig-T. 2011

Let $\omega_0, \omega \in \mathscr{P}(\mathbf{R}^{2d})$. Then $\operatorname{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

$\operatorname{Tp}_\phi(\omega_0) = \operatorname{Op}^w(\widetilde{\omega}_0)$, where $\widetilde{\omega}_0 = \omega_0 * W_{\phi,\phi} \in S^{(\omega_0)}$ and $\widetilde{\omega}_0 \asymp \omega_0$.
This gives $\operatorname{Tp}_\phi(\omega_0)$ is continuous from $M^{p,q}_{(\omega)}$ to $M^{p,q}_{(\omega/\omega_0)}$.

(2) $\vartheta = \omega_0^{\frac{1}{2}}$. Then $\operatorname{Tp}(\omega_0)$ lifts $M^2_{(\vartheta)}$ to $M^2_{(1/\vartheta)}$. Let T_1 be the inverse.

(3) By Bony-Chemin: There are $a \in S^{(\vartheta)}$ and $b \in S^{(1/\vartheta)}$ such that $\operatorname{Op}^w(a) \circ \operatorname{Op}^w(b) = \operatorname{Op}^w(a) \circ \operatorname{Op}^w(b) = \operatorname{Id}_{\mathscr{S}'}$.

(4) $\operatorname{Op}^w(c) \equiv \operatorname{Op}^w(b) \circ \operatorname{Tp}(\omega_0) \circ \operatorname{Op}^w(b)$ is bijective on L^2. Symb. calc.: $c \in S^0_{0,0}$.

(5) Since $S^0_{0,0}$ is a Wiener algebra, c_1 in $\operatorname{Op}^w(c_1) \equiv \operatorname{Op}^w(c)^{-1}$ belongs to $S^0_{0,0}$. This gives $T_1 = \operatorname{Op}^w(b) \circ \operatorname{Op}^w(c_1) \circ \operatorname{Op}^w(b) \in \operatorname{Op}^w(S^{(1/\omega_0)})$.

From these properties:

$$\operatorname{Tp}(\omega_0) : M^{p,q}_{(\omega)} \to M^{p,q}_{(\omega/\omega_0)}, \quad T_1 : M^{p,q}_{(\omega/\omega)} \to M^{p,q}_{(\omega)} \qquad (*)$$

are continuous and $\operatorname{Tp}(\omega_0) \circ T_1 = T_1 \circ \operatorname{Tp}(\omega_0) = \operatorname{Id}_{\mathscr{S}'}$.

Arguments in the proof

Gröchenig-T. 2011

Let $\omega_0, \omega \in \mathscr{P}(\mathbf{R}^{2d})$. Then $\mathrm{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

$\mathrm{Tp}_\phi(\omega_0) = \mathrm{Op}^w(\widetilde{\omega}_0)$, where $\widetilde{\omega}_0 = \omega_0 * W_{\phi,\phi} \in S^{(\omega_0)}$ and $\widetilde{\omega}_0 \asymp \omega_0$.
This gives $\mathrm{Tp}_\phi(\omega_0)$ is continuous from $M^{p,q}_{(\omega)}$ to $M^{p,q}_{(\omega/\omega_0)}$.

(2) $\vartheta = \omega_0^{\frac{1}{2}}$. Then $\mathrm{Tp}(\omega_0)$ lifts $M^2_{(\vartheta)}$ to $M^2_{(1/\vartheta)}$. Let T_1 be the inverse.

(3) By Bony-Chemin: There are $a \in S^{(\vartheta)}$ and $b \in S^{(1/\vartheta)}$ such that $\mathrm{Op}^w(a) \circ \mathrm{Op}^w(b) = \mathrm{Op}^w(a) \circ \mathrm{Op}^w(b) = \mathrm{Id}_{\mathscr{S}'}$.

(4) $\mathrm{Op}^w(c) \equiv \mathrm{Op}^w(b) \circ \mathrm{Tp}(\omega_0) \circ \mathrm{Op}^w(b)$ is bijective on L^2. Symb. calc.: $c \in S^0_{0,0}$.

(5) Since $S^0_{0,0}$ is a Wiener algebra, c_1 in $\mathrm{Op}^w(c_1) \equiv \mathrm{Op}^w(c)^{-1}$ belongs to $S^0_{0,0}$. This gives $T_1 = \mathrm{Op}^w(b) \circ \mathrm{Op}^w(c_1) \circ \mathrm{Op}^w(b) \in \mathrm{Op}^w(S^{(1/\omega_0)})$.

From these properties:

$$\mathrm{Tp}(\omega_0) : M^{p,q}_{(\omega)} \to M^{p,q}_{(\omega/\omega_0)}, \quad T_1 : M^{p,q}_{(\omega/\omega)} \to M^{p,q}_{(\omega)} \qquad (*)$$

are continuous and $\mathrm{Tp}(\omega_0) \circ T_1 = T_1 \circ \mathrm{Tp}(\omega_0) = \mathrm{Id}_{\mathscr{S}'}$. This implies that the mappings (*) are bijective and inverses to each others, giving the result.

Post-futuristic liftings

Thm. 1. Gröchenig-T. 2011 - Classic lifting

Let $\omega_0, \omega \in \mathscr{P}(\mathbf{R}^{2d})$. Then $\mathrm{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

Thm. 2. Abdeljawad-Coriasco-T. 2017 - Post-futuristic lifting

Let $\omega_0, \omega \in \mathscr{P}^0_E(\mathbf{R}^{2d})$. Then $\mathrm{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

Structure of the proof of Thm. 1.

(1) $\mathrm{Tp}_\phi(\omega_0) = \mathrm{Op}^w(\widetilde{\omega}_0)$, where $\widetilde{\omega}_0 = \omega_0 * W_{\phi,\phi} \in S^{(\omega_0)}$ and $\widetilde{\omega}_0 \asymp \omega_0$.
$\Rightarrow$ $\mathrm{Tp}_\phi(\omega_0)$ is continuous from $M^{p,q}_{(\omega)}$ to $M^{p,q}_{(\omega/\omega_0)}$.

(2) $\vartheta = \omega_0^{\frac{1}{2}}$. Then $\mathrm{Tp}(\omega_0)$ lifts $M^2_{(\vartheta)}$ to $M^2_{(1/\vartheta)}$.

(3) Bony-Chemin: There are $a \in S^{(\vartheta)}$ and $b \in S^{(1/\vartheta)}$ such that
$\mathrm{Op}^w(a) \circ \mathrm{Op}^w(b) = \mathrm{Op}^w(a) \circ \mathrm{Op}^w(b) = \mathrm{Id}_{\mathscr{S}'}$.

(4) By (2–3): $\mathrm{Op}^w(c) \equiv \mathrm{Op}^w(b) \circ \mathrm{Tp}(\omega_0) \circ \mathrm{Op}^w(b)$ is bijective on L^2, $c \in S^0_{0,0}$.

(5) Since $S^0_{0,0}$ is a Wiener algebra, c_1 in $\mathrm{Op}^w(c_1) \equiv \mathrm{Op}^w(c)^{-1}$ belongs to $S^0_{0,0}$.

Which steps work for the proof of Thm. 2?

Post-futuristic liftings

Thm. 1. Gröchenig-T. 2011 - Classic lifting

Let $\omega_0, \omega \in \mathscr{P}(\mathbf{R}^{2d})$. Then $\operatorname{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

Thm. 2. Abdeljawad-Coriasco-T. 2017 - Post-futuristic lifting

Let $\omega_0, \omega \in \mathscr{P}^0_E(\mathbf{R}^{2d})$. Then $\operatorname{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

(1) $\operatorname{Tp}_\phi(\omega_0) = \operatorname{Op}^w(\widetilde{\omega}_0)$, where $\widetilde{\omega}_0 = \omega_0 * W_{\phi,\phi} \in S^{(\omega_0)}$ and $\widetilde{\omega}_0 \asymp \omega_0$.
$\Rightarrow$ $\operatorname{Tp}_\phi(\omega_0)$ is continuous from $M^{p,q}_{(\omega)}$ to $M^{p,q}_{(\omega/\omega_0)}$.

(2) $\vartheta = \omega_0^{\frac{1}{2}}$. Then $\operatorname{Tp}(\omega_0)$ lifts $M^2_{(\vartheta)}$ to $M^2_{(1/\vartheta)}$.

(3) Bony-Chemin: There are $a \in S^{(\vartheta)}$ and $b \in S^{(1/\vartheta)}$ such that
$\operatorname{Op}^w(a) \circ \operatorname{Op}^w(b) = \operatorname{Op}^w(a) \circ \operatorname{Op}^w(b) = \operatorname{Id}_{\mathscr{S}'}$.

(4) By (2–3): $\operatorname{Op}^w(c) \equiv \operatorname{Op}^w(b) \circ \operatorname{Tp}(\omega_0) \circ \operatorname{Op}^w(b)$ is bijective on L^2, $c \in S^0_{0,0}$.

(5) Since $S^0_{0,0}$ is a Wiener algebra, c_1 in $\operatorname{Op}^w(c_1) \equiv \operatorname{Op}^w(c)^{-1}$ belongs to $S^0_{0,0}$.

Post-futuristic liftings

Thm. 1. Gröchenig-T. 2011 - Classic lifting

Let $\omega_0, \omega \in \mathscr{P}(\mathbf{R}^{2d})$. Then $\operatorname{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

Thm. 2. Abdeljawad-Coriasco-T. 2017 - Post-futuristic lifting

Let $\omega_0, \omega \in \mathscr{P}^0_E(\mathbf{R}^{2d})$. Then $\operatorname{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

(1) $\operatorname{Tp}_\phi(\omega_0) = \operatorname{Op}^w(\widetilde{\omega}_0)$, where $\widetilde{\omega}_0 = \omega_0 * W_{\phi,\phi} \in S^{(\omega_0)}$ and $\widetilde{\omega}_0 \asymp \omega_0$.
$\Rightarrow$ $\operatorname{Tp}_\phi(\omega_0)$ is continuous from $M^{p,q}_{(\omega)}$ to $M^{p,q}_{(\omega/\omega_0)}$.

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Step (1) works fine also when $\omega, \omega_0 \in \mathscr{P}^0_E$, provided $\phi \in \mathcal{S}_1$.

Then (1) holds with $\widetilde{\omega}_0 \in \Gamma^{(\omega_0)}_1$.

Post-futuristic liftings

Thm. 1. Gröchenig-T. 2011 - Classic lifting

Let $\omega_0, \omega \in \mathscr{P}(\mathbf{R}^{2d})$. Then $\operatorname{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

Thm. 2. Abdeljawad-Coriasco-T. 2017 - Post-futuristic lifting

Let $\omega_0, \omega \in \mathscr{P}^0_E(\mathbf{R}^{2d})$. Then $\operatorname{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

(1) $\operatorname{Tp}_\phi(\omega_0) = \operatorname{Op}^w(\widetilde{\omega}_0)$, where $\widetilde{\omega}_0 = \omega_0 * W_{\phi,\phi} \in S^{(\omega_0)}$ and $\widetilde{\omega}_0 \asymp \omega_0$.
$\Rightarrow$ $\operatorname{Tp}_\phi(\omega_0)$ is continuous from $M^{p,q}_{(\omega)}$ to $M^{p,q}_{(\omega/\omega_0)}$.

(2) $\vartheta = \omega_0^{\frac{1}{2}}$. Then $\operatorname{Tp}(\omega_0)$ lifts $M^2_{(\vartheta)}$ to $M^2_{(1/\vartheta)}$.

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(5) Since $S^0_{0,0}$ is a Wiener algebra, c_1 in $\operatorname{Op}^w(c_1) \equiv \operatorname{Op}^w(c)^{-1}$ belongs to $S^0_{0,0}$.

Post-futuristic liftings

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Thm. 2. Abdeljawad-Coriasco-T. 2017 - Post-futuristic lifting

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Step (2) works fine with the same arguments.

Post-futuristic liftings

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Let $\omega_0, \omega \in \mathscr{P}^0_E(\mathbf{R}^{2d})$. Then $\mathrm{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

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Post-futuristic liftings

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Thm. 2. Abdeljawad-Coriasco-T. 2017 - Post-futuristic lifting

Let $\omega_0, \omega \in \mathscr{P}^0_E(\mathbf{R}^{2d})$. Then $\mathrm{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

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$\Rightarrow$ $\mathrm{Tp}_\phi(\omega_0)$ is continuous from $M^{p,q}_{(\omega)}$ to $M^{p,q}_{(\omega/\omega_0)}$.

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(5) Since $S^0_{0,0}$ is a Wiener algebra, c_1 in $\mathrm{Op}^w(c_1) \equiv \mathrm{Op}^w(c)^{-1}$ belongs to $S^0_{0,0}$.

Step (3) needs to be replaced by:

Post-futuristic liftings

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Let $\omega_0, \omega \in \mathscr{P}(\mathbf{R}^{2d})$. Then $\operatorname{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

Thm. 2. Abdeljawad-Coriasco-T. 2017 - Post-futuristic lifting

Let $\omega_0, \omega \in \mathscr{P}^0_E(\mathbf{R}^{2d})$. Then $\operatorname{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

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$\Rightarrow$ $\operatorname{Tp}_\phi(\omega_0)$ is continuous from $M^{p,q}_{(\omega)}$ to $M^{p,q}_{(\omega/\omega_0)}$.

(2) $\vartheta = \omega_0^{\frac{1}{2}}$. Then $\operatorname{Tp}(\omega_0)$ lifts $M^2_{(\vartheta)}$ to $M^2_{(1/\vartheta)}$.

(3) Bony-Chemin: There are $a \in S^{(\vartheta)}$ and $b \in S^{(1/\vartheta)}$ such that
$\operatorname{Op}^w(a) \circ \operatorname{Op}^w(b) = \operatorname{Op}^w(a) \circ \operatorname{Op}^w(b) = \operatorname{Id}_{\mathscr{S}'}$.

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(5) Since $S^0_{0,0}$ is a Wiener algebra, c_1 in $\operatorname{Op}^w(c_1) \equiv \operatorname{Op}^w(c)^{-1}$ belongs to $S^0_{0,0}$.

Step (3) needs to be replaced by:

(3)′ There are $a \in \Gamma_1^{(\vartheta)}$ and $b \in \Gamma_1^{(1/\vartheta)}$ such that
$\operatorname{Op}^w(a) \circ \operatorname{Op}^w(b) = \operatorname{Op}^w(a) \circ \operatorname{Op}^w(b) = \operatorname{Id}_{\mathcal{S}_1'}$.

Post-futuristic liftings

Thm. 1. Gröchenig-T. 2011 - Classic lifting

Let $\omega_0, \omega \in \mathscr{P}(\mathbf{R}^{2d})$. Then $\mathrm{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

Thm. 2. Abdeljawad-Coriasco-T. 2017 - Post-futuristic lifting

Let $\omega_0, \omega \in \mathscr{P}^0_E(\mathbf{R}^{2d})$. Then $\mathrm{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

(1) $\mathrm{Tp}_\phi(\omega_0) = \mathrm{Op}^w(\widetilde{\omega}_0)$, where $\widetilde{\omega}_0 = \omega_0 * W_{\phi,\phi} \in S^{(\omega_0)}$ and $\widetilde{\omega}_0 \asymp \omega_0$.
$\Rightarrow$ $\mathrm{Tp}_\phi(\omega_0)$ is continuous from $M^{p,q}_{(\omega)}$ to $M^{p,q}_{(\omega/\omega_0)}$.

(2) $\vartheta = \omega_0^{\frac{1}{2}}$. Then $\mathrm{Tp}(\omega_0)$ lifts $M^2_{(\vartheta)}$ to $M^2_{(1/\vartheta)}$.

(3) Bony-Chemin: There are $a \in S^{(\vartheta)}$ and $b \in S^{(1/\vartheta)}$ such that
$\mathrm{Op}^w(a) \circ \mathrm{Op}^w(b) = \mathrm{Op}^w(a) \circ \mathrm{Op}^w(b) = \mathrm{Id}_{\mathscr{S}'}$.

(4) By (2–3): $\mathrm{Op}^w(c) \equiv \mathrm{Op}^w(b) \circ \mathrm{Tp}(\omega_0) \circ \mathrm{Op}^w(b)$ is bijective on L^2, $c \in S^0_{0,0}$.

(5) Since $S^0_{0,0}$ is a Wiener algebra, c_1 in $\mathrm{Op}^w(c_1) \equiv \mathrm{Op}^w(c)^{-1}$ belongs to $S^0_{0,0}$.

(3)′ There are $a \in \Gamma_1^{(\vartheta)}$ and $b \in \Gamma_1^{(1/\vartheta)}$ such that
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Post-futuristic liftings

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Thm. 2. Abdeljawad-Coriasco-T. 2017 - Post-futuristic lifting

Let $\omega_0, \omega \in \mathscr{P}^0_E(\mathbf{R}^{2d})$. Then $\mathrm{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

(1) $\mathrm{Tp}_\phi(\omega_0) = \mathrm{Op}^w(\widetilde{\omega}_0)$, where $\widetilde{\omega}_0 = \omega_0 * W_{\phi,\phi} \in S^{(\omega_0)}$ and $\widetilde{\omega}_0 \asymp \omega_0$.
$\Rightarrow$ $\mathrm{Tp}_\phi(\omega_0)$ is continuous from $M^{p,q}_{(\omega)}$ to $M^{p,q}_{(\omega/\omega_0)}$.

(2) $\vartheta = \omega_0^{\frac{1}{2}}$. Then $\mathrm{Tp}(\omega_0)$ lifts $M^2_{(\vartheta)}$ to $M^2_{(1/\vartheta)}$.

(3) Bony-Chemin: There are $a \in S^{(\vartheta)}$ and $b \in S^{(1/\vartheta)}$ such that
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(4) By (2–3): $\mathrm{Op}^w(c) \equiv \mathrm{Op}^w(b) \circ \mathrm{Tp}(\omega_0) \circ \mathrm{Op}^w(b)$ is bijective on L^2, $c \in S^0_{0,0}$.

(5) Since $S^0_{0,0}$ is a Wiener algebra, c_1 in $\mathrm{Op}^w(c_1) \equiv \mathrm{Op}^w(c)^{-1}$ belongs to $S^0_{0,0}$.

(3)′ There are $a \in \Gamma_1^{(\vartheta)}$ and $b \in \Gamma_1^{(1/\vartheta)}$ such that
$\mathrm{Op}^w(a) \circ \mathrm{Op}^w(b) = \mathrm{Op}^w(a) \circ \mathrm{Op}^w(b) = \mathrm{Id}_{\mathcal{S}_1'}$.

Step (4) is working fine with $\Gamma^0_{1;0}$ in place of $S^0_{0,0}$, provided (3)′ has been proved.

Post-futuristic liftings

Thm. 1. Gröchenig-T. 2011 - Classic lifting

Let $\omega_0, \omega \in \mathscr{P}(\mathbf{R}^{2d})$. Then $\operatorname{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

Thm. 2. Abdeljawad-Coriasco-T. 2017 - Post-futuristic lifting

Let $\omega_0, \omega \in \mathscr{P}^0_E(\mathbf{R}^{2d})$. Then $\operatorname{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

(1) $\operatorname{Tp}_\phi(\omega_0) = \operatorname{Op}^w(\widetilde{\omega}_0)$, where $\widetilde{\omega}_0 = \omega_0 * W_{\phi,\phi} \in S^{(\omega_0)}$ and $\widetilde{\omega}_0 \asymp \omega_0$.
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(2) $\vartheta = \omega_0^{\frac{1}{2}}$. Then $\operatorname{Tp}(\omega_0)$ lifts $M^2_{(\vartheta)}$ to $M^2_{(1/\vartheta)}$.

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(4) By (2–3): $\operatorname{Op}^w(c) \equiv \operatorname{Op}^w(b) \circ \operatorname{Tp}(\omega_0) \circ \operatorname{Op}^w(b)$ is bijective on L^2, $c \in S^0_{0,0}$.

(5) Since $S^0_{0,0}$ is a Wiener algebra, c_1 in $\operatorname{Op}^w(c_1) \equiv \operatorname{Op}^w(c)^{-1}$ belongs to $S^0_{0,0}$.

(3)′ There are $a \in \Gamma_1^{(\vartheta)}$ and $b \in \Gamma_1^{(1/\vartheta)}$ such that
$\operatorname{Op}^w(a) \circ \operatorname{Op}^w(b) = \operatorname{Op}^w(a) \circ \operatorname{Op}^w(b) = \operatorname{Id}_{\mathcal{S}_1'}$.

Post-futuristic liftings

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Thm. 2. Abdeljawad-Coriasco-T. 2017 - Post-futuristic lifting

Let $\omega_0, \omega \in \mathscr{P}^0_E(\mathbf{R}^{2d})$. Then $\operatorname{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

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$\Rightarrow$ $\operatorname{Tp}_\phi(\omega_0)$ is continuous from $M^{p,q}_{(\omega)}$ to $M^{p,q}_{(\omega/\omega_0)}$.

(2) $\vartheta = \omega_0^{\frac{1}{2}}$. Then $\operatorname{Tp}(\omega_0)$ lifts $M^2_{(\vartheta)}$ to $M^2_{(1/\vartheta)}$.

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(5) Since $S^0_{0,0}$ is a Wiener algebra, c_1 in $\operatorname{Op}^w(c_1) \equiv \operatorname{Op}^w(c)^{-1}$ belongs to $S^0_{0,0}$.

(3)′ There are $a \in \Gamma^{(\vartheta)}_1$ and $b \in \Gamma^{(1/\vartheta)}_1$ such that
$\operatorname{Op}^w(a) \circ \operatorname{Op}^w(b) = \operatorname{Op}^w(a) \circ \operatorname{Op}^w(b) = \mathrm{Id}_{\mathcal{S}'_1}$.

Step (5) also works with $\Gamma^0_{1;0}$ in place of $S^0_{0,0}$. Again it is essential to prove (3)′.

Post-futuristic liftings

Thm. 1. Gröchenig-T. 2011 - Classic lifting

Let $\omega_0, \omega \in \mathscr{P}(\mathbf{R}^{2d})$. Then $\mathrm{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

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Let $\omega_0, \omega \in \mathscr{P}^0_E(\mathbf{R}^{2d})$. Then $\mathrm{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

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Post-futuristic liftings

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Thm. 2. Abdeljawad-Coriasco-T. 2017 - Post-futuristic lifting

Let $\omega_0, \omega \in \mathscr{P}^0_E(\mathbf{R}^{2d})$. Then $\mathrm{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\omega_0)}(\mathbf{R}^d)$.

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(5) Since $S^0_{0,0}$ is a Wiener algebra, c_1 in $\mathrm{Op}^w(c_1) \equiv \mathrm{Op}^w(c)^{-1}$ belongs to $S^0_{0,0}$.

(3)′ There are $a \in \Gamma_1^{(\vartheta)}$ and $b \in \Gamma_1^{(1/\vartheta)}$ such that
$\mathrm{Op}^w(a) \circ \mathrm{Op}^w(b) = \mathrm{Op}^w(a) \circ \mathrm{Op}^w(b) = \mathrm{Id}_{\mathcal{S}_1'}$.

After these observations: **How to make (3)′ ?**

Some comments to the proof of $(3)'$

Thm. (Abdeljawad-Coriasco-T. 2017)

Let $\omega \in \mathscr{P}_E^0(\mathbf{R}^{2d})$. Then there are $a \in \Gamma_1^{(\omega)}(\mathbf{R}^{2d})$ and $b \in \Gamma_1^{(1/\omega)}(\mathbf{R}^{2d})$ such that

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The result is reached by expanding the theory of ΨDO with symbols in $\Gamma_{s;0}(\mathbf{R}^{2d})$ and $\Gamma_s^{(\omega)}(\mathbf{R}^{2d})$ in the lines of Bony-Chemin approaches. For example:

Some comments to the proof of (3)′

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- **Reduction to confined symbols:** If $a \in \Gamma_1^{(\omega)}(\mathbf{R}^{2d})$, $b \in \Gamma_1^{(\vartheta)}(\mathbf{R}^{2d})$, and $\phi_Y = \phi(\,\cdot\, - Y)$ and $\psi_Z = \psi(\,\cdot\, - Z)$, are suitable, then

$$|D_X^\alpha((\phi_Y a)\#(\psi_Z b))(X)| \lesssim h^{|\alpha|}\alpha! e^{-r(|X-Y|+|X-Z|)}\omega(X)\vartheta(X), \qquad (*)$$

for some $r, h > 0$.

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By integrating (*) w.r.t. Y, Z we get

$$|D_X^\alpha(a\#b)(X)| \lesssim h^{|\alpha|}\alpha!\omega(X)\vartheta(X)$$

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- Deducing Beals-characterisation of $\operatorname{Op}^w(\Gamma_1^{(\omega)})$ (suitable estimates of powers of commutators of such operator with operators with linear symbols).

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- Solving a symbol valued evolution equation:

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- Deducing Beals-characterisation of $\operatorname{Op}^w(\Gamma_1^{(\omega)})$

- Solving a symbol valued evolution equation:

$$(\partial_t a)(t,\,\cdot\,) = (\log \omega)\# a(t,\,\cdot\,), \quad a(0,\,\cdot\,) = a_0.$$

  The choice $a_0 = 1$ leads to (3)′.

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Let $p,q\in(0,\infty]$, $\omega\in\mathscr{P}_E^0(\mathbf{R}^{2d})$, $m,\mu\geq 1$ are integers, $r,\rho\geq 0$ are real and

$$\omega_0(x,\xi)=(1+|x|^{2m})^{\frac{r}{2m}}+(1+|\xi|^{2\mu})^{\frac{\rho}{2\mu}}.$$

Then $\operatorname{Tp}(\omega_0)$ and $\operatorname{Op}^w(\omega_0)$ are bijective mappings from $M^{p,q}_{(\omega)}$ to $M^{p,q}_{(\omega/\omega_0)}$.

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By combining the recent results lifting results with some Fredholm theory we get

Let $p, q \in (0, \infty]$, $\omega \in \mathscr{P}_E^0(\mathbf{R}^{2d})$, $m, \mu \geq 1$ are integers, $r, \rho \geq 0$ are real and

$$\omega_0(x, \xi) = (1 + |x|^{2m})^{\frac{r}{2m}} + (1 + |\xi|^{2\mu})^{\frac{\rho}{2\mu}}.$$

Then $\mathrm{Tp}(\omega_0)$ and $\mathrm{Op}^w(\omega_0)$ are bijective mappings from $M^{p,q}_{(\omega)}$ to $M^{p,q}_{(\omega/\omega_0)}$.

Let $p, q \in (0, \infty]$, $\omega \in \mathscr{P}_E^0(\mathbf{R}^{2d})$, $m, \mu \geq 1$ are integers, $0 \leq r, \rho < 1$ and

$$\omega_0(x, \xi) = \exp\left((1 + |x|^{2m})^{\frac{r}{2m}} + (1 + |\xi|^{2\mu})^{\frac{\rho}{2\mu}}\right).$$

Then $\mathrm{Tp}(\omega_0)$ is bijective and $\mathrm{Op}^w(\omega_0)$ has index zero as mappings from $M^{p,q}_{(\omega)}$ to $M^{p,q}_{(\omega/\omega_0)}$.

Thank you for your attention !!