Very weak solutions vs Strong singularities

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Abstract

We consider two physical models, the fractional wave equation [1] with a mass term and the fractional Schrödinger equation [2] with a potential. In both equations the coefficients depend on the spatial variables and are assumed to be singular. Delta like or even higher-order singularities are allowed. By using regularising techniques, we introduce a family of 'weakened' solutions, calling them very weak solutions [3]. The existence, uniqueness and consistency results are proved in an appropriate sense. Numerical experiments are done. The appearance of a wall effect for the singular masses of the strength of δ^2 is observed for the wave equation and a particles accumulating effect for the Schrödinger equation.

Equations

For $s > 0$ and $(t,x) \in (0,T] \times \mathbb{R}^d$, we consider the Cauchy problems

$$\begin{cases} u_{tt}(t,x) + (-\Delta)^s u(t,x) + m(x)u(t,x) = 0, \\ u(0,x) = f(x), \quad u_t(0,x) = g(x), \end{cases} \tag{FWE}$$

and

$$\begin{cases} iu_t(t,x) + (-\Delta)^s u(t,x) + p(x)u(t,x) = 0, \\ u(0,x) = h(x), \end{cases} \tag{FSE}$$

Objectives

- To prove the very weak well posedness of the Cauchy problems (FWE) and (FSE).
- To prove the consistency with classical theory.
- To study numerically the behaviour of very weak solutions to (FWE) and (FSE) near the singularities of the coefficients.

References

[1] A. Altybay, M. Ruzhansky, M. E. Sebih, N. Tokmagambetov. Fractional Klein-Gordon equation with strongly singular mass term. *arxiv*:2004.10145 (2020).

[2] A. Altybay, M. Ruzhansky, M. E. Sebih, N. Tokmagambetov. Fractional Schrödinger Equations with potentials of higher-order singularities. *arxiv*:2004.10182 (2020).

[3] C. Garetto, M. Ruzhansky. Hyperbolic second order equations with non-regular time dependent coefficients. *Arch. Rational Mech. Anal.*, 217 (2015).

Fundamental lemmas

We make use of the following notation:

$$\|u(t,\cdot)\|_{FWE} = \|u(t,\cdot)\|_{H^s} + \|\partial_t u(t,\cdot)\|_{L^2}.$$

When the coefficients are regular enough, we have

Lemma 1. *Let $s > 0$. Suppose that $m \in L^\infty(\mathbb{R}^d)$ and $m \geq 0$ and that $f \in H^s(\mathbb{R}^d)$ and $g \in L^2(\mathbb{R}^d)$. Then, there is a unique solution $u \in C([0,T], H^s(\mathbb{R}^d)) \cap C^1([0,T], L^2(\mathbb{R}^d))$ to (FWE), and it satisfies the estimate*

$$\|u(t,\cdot)\|_{FWE} \lesssim (1 + \|m\|_{L^\infty})^{\frac{1}{2}} \left[\|g\|_{L^2} + \|f\|_{H^s}\right]. \tag{1}$$

Lemma 2. *Let $s > 0$. Suppose that $p \in L^\infty(\mathbb{R}^d)$ is non-negative and assume that $h \in H^s(\mathbb{R}^d)$. Then the estimates*

$$\|u(t,\cdot)\|_{H^s} \lesssim (1 + \|p\|_{L^\infty}) \|h\|_{H^s}, \tag{2}$$

$$\|u(t,\cdot)\|_{L^2} = \|h\|_{L^2}, \tag{3}$$

hold for the unique solution $u \in C([0,T], H^s)$ to the Cauchy problem (FSE).

Definitions/Assumptions

We regularise the coefficients in FWE and FSE by convolution with a suitable mollifier ψ and obtain $m_\varepsilon(x) = m * \psi_\varepsilon(x)$, $p_\varepsilon(x) = p * \psi_\varepsilon(x)$ and $h_\varepsilon(x) = h * \psi_\varepsilon(x)$, where $\psi_\varepsilon(x) = \varepsilon^{-1}\psi(x/\varepsilon)$ and $\varepsilon \in (0,1]$. The function ψ is a Friedrichs-mollifier.

Definition 1. (Moderateness)

- *A net of functions $(f_\varepsilon)_\varepsilon$ is said to be X-moderate, if there exist $N \in \mathbb{N}_0$ and $c > 0$ such that*

  $$\|f_\varepsilon\|_X \leq c\varepsilon^{-N},$$

  where the function space X is either $L^\infty(\mathbb{R}^d)$ or $H^s(\mathbb{R}^d)$.

- *The net of functions $(u_\varepsilon)_\varepsilon$ from $C([0,T], H^s)$ is said to be C-moderate, if there exist $N \in \mathbb{N}_0$ and $c > 0$ such that*

  $$\sup_{t\in[0,T]} \|u_\varepsilon(t,\cdot)\|_{H^s} \leq c\varepsilon^{-N}.$$

- *The net of functions $(u_\varepsilon)_\varepsilon$ from $C([0,T], H^s) \cap C^1([0,T], L^2)$ is said to be C^1-moderate, if there exist $N \in \mathbb{N}_0$ and $c > 0$ such that*

  $$\sup_{t\in[0,T]} \|u_\varepsilon(t,\cdot)\|_{FWE} \leq c\varepsilon^{-N}.$$

...Definitions/Assumptions

Assumptions . *We assume that:*

(A1) *m and p are non-negative.*

(A2) *$(m_\varepsilon)_\varepsilon$ and $(p_\varepsilon)_\varepsilon$ are L^∞-moderate and that $(h_\varepsilon)_\varepsilon$ is H^s-moderate.*

Definition 2. (Very weak solution)

- *Let $(f,g) \in H^s(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. Then the net $(u_\varepsilon)_\varepsilon \in C([0,T], H^s(\mathbb{R}^d)) \cap C^1([0,T], L^2(\mathbb{R}^d))$ is a very weak solution of order s to the Cauchy problem (FWE) if there exists an L^∞-moderate regularisation $(m_\varepsilon)_\varepsilon$ of the coefficient m such that $(u_\varepsilon)_\varepsilon$ solves the regularized problem*

  $$\begin{cases} \partial_t^2 u_\varepsilon(t,x) + (-\Delta)^s u_\varepsilon(t,x) + m_\varepsilon(x)u_\varepsilon(t,x) = 0, \\ u_\varepsilon(0,x) = f(x), \partial_t u_\varepsilon(0,x) = g(x), \quad x \in \mathbb{R}^d, \end{cases}$$

  for all $\varepsilon \in (0,1]$, and is C^1-moderate.

- *The net $(v_\varepsilon)_\varepsilon \in C([0,T], H^s)$ is said to be a very weak solution of order s to the Cauchy problem (FSE) if there exist an L^∞-moderate regularisation of the coefficient p and H^s-moderate regularisation of h such that $(v_\varepsilon)_\varepsilon$ solves the regularized problem*

  $$\begin{cases} i\partial_t v_\varepsilon(t,x) + (-\Delta)^s v_\varepsilon(t,x) + p_\varepsilon(x)v_\varepsilon(t,x) = 0, \\ v_\varepsilon(0,x) = h_\varepsilon(x), \end{cases}$$

  for all $\varepsilon \in (0,1]$, and is C-moderate.

Very weak well posedness

Theorem 1. *Let $s > 0$ and assume **(A1)** and **(A2)** to be satisfied. Then the Cauchy problems (FWE) and (FSE) have unique very weak solutions of order s.*

...Very weak well posedness

The uniqueness is proved in the following sense.

Definition 3. *We say that the Cauchy problem (FWE) has a unique very weak solution, if for all nets of regularisations $(m_\varepsilon)_\varepsilon$ and $(\tilde{m}_\varepsilon)_\varepsilon$, of m, satisfying $\|m_\varepsilon - \tilde{m}_\varepsilon\|_{L^\infty} \leq C_k\varepsilon^k$ for all $k > 0$, it follows that*

$$\|u_\varepsilon(t,\cdot) - \tilde{u}_\varepsilon(t,\cdot)\|_{L^2} \leq C_N\varepsilon^N$$

for all $N > 0$ and $t \in [0,T]$, where $(u_\varepsilon)_\varepsilon$ and $(\tilde{u}_\varepsilon)_\varepsilon$ are the families of solutions corresponding to $(m_\varepsilon)_\varepsilon$ and $(\tilde{m}_\varepsilon)_\varepsilon$.

Definition 4. *We say that the Cauchy problem (FSE) has a unique very weak solution, if for all nets of regularisations $(p_\varepsilon)_\varepsilon$, $(\tilde{p}_\varepsilon)_\varepsilon$, $(u_{0,\varepsilon})_\varepsilon$ and $(\tilde{u}_{0,\varepsilon})_\varepsilon$ of p and u_0, satisfying $\|p_\varepsilon - \tilde{p}_\varepsilon\|_{L^\infty} \leq C_k\varepsilon^k$ for all $k > 0$ and $\|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} \leq C_l\varepsilon^l$ for all $l > 0$, we have*

$$\|u_\varepsilon(t,\cdot) - \tilde{u}_\varepsilon(t,\cdot)\|_{L^2} \leq C_N\varepsilon^N$$

for all $N > 0$, where $(u_\varepsilon)_\varepsilon$ and $(\tilde{u}_\varepsilon)_\varepsilon$ are the families of solutions to the corresponding regularized Cauchy problems.

Consistency

We prove that the V.W. solutions to (FWE) and (FSE) recapture the classical ones when they exist.

Theorem 2. *Let $s > 0$. Let $(m,f,g) \in L^\infty(\mathbb{R}^d) \times H^s(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ and $(p,h) \in L^\infty(\mathbb{R}^d) \times H^s(\mathbb{R}^d)$. Let $(u_\varepsilon)_\varepsilon$ and $(v_\varepsilon)_\varepsilon$ be very weak solutions of (FWE)and (FSE) respectively. Then for any regularising families of the coefficients and the Cauchy data in (FWE) and (FSE), the nets $(u_\varepsilon)_\varepsilon$ and $(v_\varepsilon)_\varepsilon$ converge in L^2 as $\varepsilon \to 0$ to the unique classical solutions of the Cauchy problems (FWE) and (FSE) respectively.*

Numerical experiments

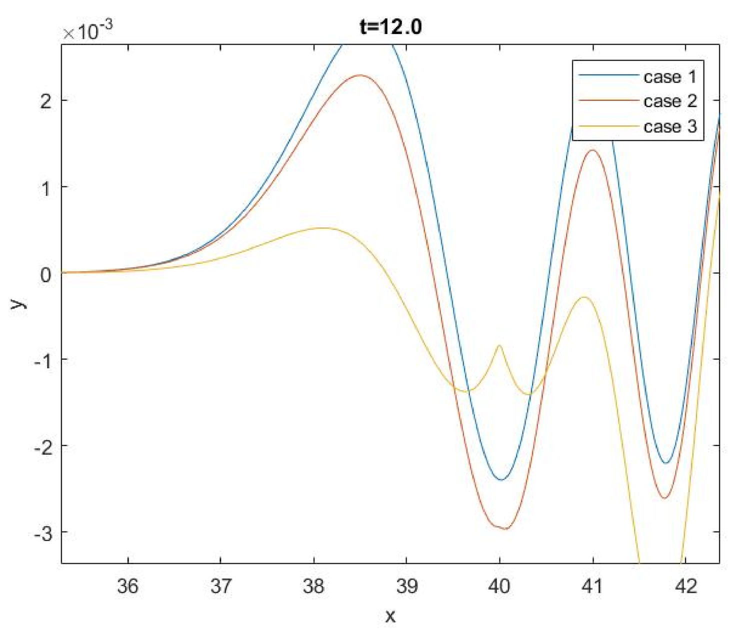

Figure 1: We analyse the behaviour of the solution to FWE for different masses. When $m(x) = \delta(x-40)$, i.e. $m_\varepsilon(x) = \varphi_\varepsilon(x-40)$, we observe the appearance of a wall effect.

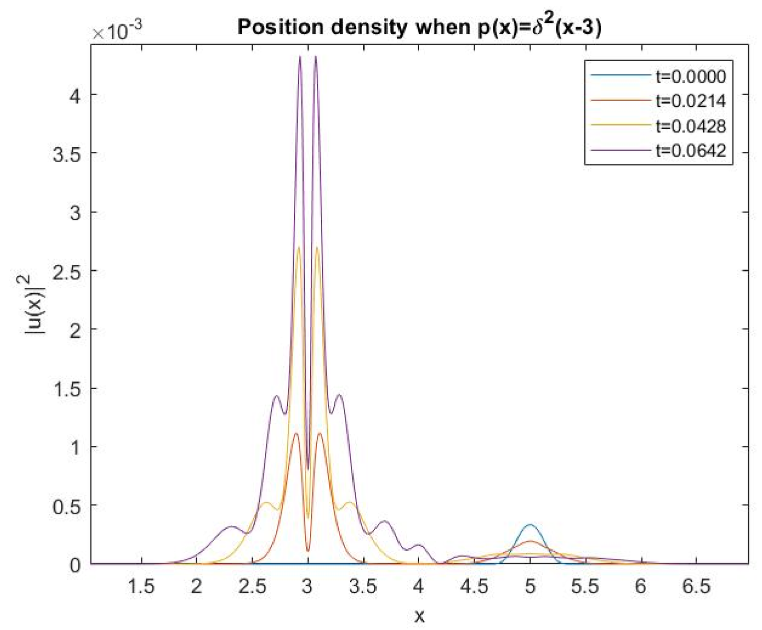

Figure 2: We analyse the behaviour of the solution to FSE for a δ^2-like potential for different times. We observe a particles accumulating effect.