**Abstract**

We consider two physical models, the fractional wave equation [1] with a mass term and the fractional Schrödinger equation [2] with a potential. In both equations the coefficients depend on the spatial variables and are assumed to be singular. Delta like or even higher-order singularities are allowed. By using regularising techniques, we introduce a family of ‘weakened’ solutions, calling them very weak solutions [3]. The existence, uniqueness and consistency results are proved in an appropriate sense. Numerical experiments are done. The appearance of a wall effect for the singular masses of the strength of $\delta^2$ is observed for the wave equation and particles accumulating effect for the Schrödinger equation.

**Equations**

For $s > 0$ and $(t, x) \in (0, T) \times \mathbb{R}^d$, we consider the Cauchy problems

\[
\begin{aligned}
\left\{ 
\begin{array}{l}
u_t(t,x)+(\Delta)^s \nu(t,x)+m(t,x)\nu(t,x)=0, \\
u(0,x)=f(x), \\
u(0,x)=g(x),
\end{array}
\right.
\end{aligned}
\]

and

\[
\begin{aligned}
u_t(t,x)+(\Delta)^s \nu(t,x)+p(x)\nu(t,x)=0, \\
u(0,x)=h(x),
\end{aligned}
\]

for the fractional wave equation (FWE) and the fractional Schrödinger equation (FSE), respectively.

**Objectives**

- To prove the very weak well posedness of the Cauchy problems (FWE) and (FSE).
- To prove the consistency with classical theory.
- To study numerically the behaviour of very weak solutions to (FWE) and (FSE) near the singularities of the coefficients.

**Fundamental Lemmas**

We make use of the following notation:

\[
\|u(t)\|_{L^2} = \|u(t)\|_{L^2}\text{ and }\|\delta \partial_t u(t)\|_{L^2}.
\]

When the coefficients are regular enough, we have

**Lemma 1.** Let $s > 0$. Suppose that $m \in L^\infty(\mathbb{R}^d)$ and $m \geq 0$ and that $f \in H^s(\mathbb{R}^d)$ and $g \in L^2(\mathbb{R}^d)$. Then, there is a unique solution $u \in C([0,T], H^s(\mathbb{R}^d)) \cap C^1([0,T], L^2(\mathbb{R}^d))$ to (FWE), and it satisfies

\[
\|\delta \partial_t u(t)\|_{L^2} \leq (1 + m(t,x)) \|f\|_{L^2} + \|g\|_{L^2}.
\]

**Lemma 2.** Let $s > 0$. Suppose that $p \in L^\infty(\mathbb{R}^d)$ is non-negative and assume that $\epsilon \in H^s(\mathbb{R}^d)$. Then the estimates

\[
\|u(t)\|_{H^s(\mathbb{R}^d)} \leq (1 + |p|_{L^\infty(\mathbb{R}^d)}) \|\epsilon\|_{H^s(\mathbb{R}^d)},
\]

for all $t \in [0, T]$, and $C^l$-moderate. hold for the unique solution $u \in C([0,T], H^s) \cap C^1([0,T], L^2(\mathbb{R}^d))$ to the Cauchy problems (FSE).

**Definitions/Assumptions**

We regularise the coefficients in FWE and FSE by convolution with a suitable mollifier $\psi$ and obtain

\[
m_\epsilon(x) = m + \epsilon^2 \psi(x),
\]

\[
p_\epsilon(x) = p \ast \psi(x)
\]

and

\[
h_\epsilon(x) = h \ast \psi(x), \quad \psi \in \mathcal{D}(\mathbb{R}^d).
\]

The function $\psi$ is a Friedricts-mollifier.

**Definition 1.** (Moderatness) A net of functions $(f_\epsilon)_\epsilon$ is said to be $C^l$-moderate, if there exist $N \in \mathbb{N}$ and $\epsilon > 0$ such that

\[
\|f_\epsilon\|_{S^N} \leq \epsilon^{N},
\]

where the function space $S^N$ is either $L^\infty(\mathbb{R}^d)$ or $H^s(\mathbb{R}^d)$.

**Theorem 1.** Let $s > 0$ and assume (A1) and (A2) to be satisfied. Then the Cauchy problems (FWE) and (FSE) have unique very weak solutions of order $s$.

**Numerical experiments**

Figure 1: We analyse the behaviour of the solution to FWE for different masses. When $m(\cdot) = \varphi(\cdot) - 30$, i.e. $m(\cdot) = \varphi(\cdot) - 30$, we observe the appearance of a wall effect.

Figure 2: We analyse the behaviour of the solution to FSE for a $\delta^2$-like potential for different times. We observe a particles accumulating effect.

**References**


**Very weak solutions vs Strong singularities**

A. Altybay, M. Ruzhansky, M. Sebih and N. Tokmagambetov

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**Definitions of Assumptions**

Assumptions. We assume that:

(A1) $m$ and $p$ are non-negative.

(A2) $(m_\epsilon)$ and $(p_\epsilon)$ are $L^\infty$-moderate and that $(h_\epsilon)$ is $H^s$-moderate.

Definition 2. (Very weak solution) Let $(f, g) \in H^s(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. Then the net $(u_\epsilon) \in C([0,T], H^s(\mathbb{R}^d)) \cap C^1([0,T], L^2(\mathbb{R}^d))$ is a very weak solution of order $s$ to the Cauchy problem (FWE) if there exists an $L^\infty$-moderate regularisation $(m_\epsilon)$ of the coefficients $m$ such that $(u_\epsilon)$ solves the regularised problem

\[
\begin{aligned}
\partial_t u_\epsilon(t,x) + \Delta^s u_\epsilon(t,x) + m_\epsilon(t,x)u_\epsilon(t,x) = 0,
\quad u_\epsilon(0,x) = f(x),
\quad u_\epsilon(0,x) = g(x),
\end{aligned}
\]

for all $\epsilon \in (0, 1)$, and is $C^l$-moderate.

**Consistency**

We prove that the V.W. solutions to (FWE) and (FSE) recapture the classical ones when they exist.

Theorem 2. Let $s > 0$. Let $(m, f, g) \in L^\infty(\mathbb{R}^d) \times H^s(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ and $(p, h) \in L^\infty(\mathbb{R}^d) \times H^s(\mathbb{R}^d)$. Let $(u_\epsilon)$ and $(v_\epsilon)$ be very weak solutions of (FWE) and (FSE) respectively. Then for any regularising families of coefficients and the Cauchy data in (FWE) and (FSE), the nets $(u_\epsilon)$ and $(v_\epsilon)$ converge in $L^2$ as $\epsilon \to 0$ to the unique classical solutions of the Cauchy problems (FWE) and (FSE) respectively.

**Very weak well posedness**

The uniqueness is proved in the following sense.

Definition 3. We say that the Cauchy problem (FWE) has a unique very weak solution, if for all nets of regularisations $(m_\epsilon)$ and $(\tilde{m}_\epsilon)$, of $m$, satisfying $\|m_\epsilon - \tilde{m}_\epsilon\|_{L^\infty} \leq C_N \epsilon^N$ for all $k > 0$, it follows that

\[
\|u_\epsilon(t) - \tilde{u}_\epsilon(t)\|_{L^2} \leq C_N \epsilon^N
\]

for all $N > 0$ and $t \in (0, T]$, where $(u_\epsilon)$ and $(\tilde{u}_\epsilon)$ are the families of solutions corresponding to $(m_\epsilon)$ and $(\tilde{m}_\epsilon)$.