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The Heat Semigroup of the Twisted Laplacian on the Non-Isotropic Heisenberg Group with Multi-Dimensional Center

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An Epilog

This is joint work with my Ph.D. student, Ms. Shengwen Yang, of York University. In this short talk, I'll describe some results in her Ph.D. dissertation. I have chosen the heat semigroup to be presented in this talk.

References

- M.W. Wong, The heat equation for the Hermite operator on the Heisenberg group, *Hokkaido Mathematical Journal* (**34**) 2005, 393–404.
- M. W. Wong, Weyl transforms, the heat kernel and Green function of a degenerate elliptic operator, *Annals of Global Analysis and Geometry* (**28**) 2005, 271–283.
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A Heisenberg Group with Higher Complexity

Let $B_1, B_2, \ldots, B_m \in O(n, \mathbb{R})$ be such that

$$B_j^{-1} B_k = -B_k^{-1} B_j, \quad j \neq k.$$

We let

$$B_\lambda = \sum_{j=1}^{m} \lambda_j B_j, \quad \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{R}^m \setminus \{0\}$$

and suppose that

$$\det B_\lambda \neq 0.$$

A Heisenberg Group with Higher Complexity

The group $\mathbb{G}$ is the set $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ with the group law $\cdot_{\mathbb{G}}$ given by

$$(z,t) \cdot_{\mathbb{G}} (z',t') = \left(z + z', t + t' + \frac{1}{2}[z,z']\right)$$

for all $(z,t),(z',t') \in \mathbb{G}$, where $[z,z'] \in \mathbb{R}^m$ is given by

$$[z,z']_j = x' \cdot B_j y - x \cdot B_j y', \quad j = 1,2,\ldots,m.$$

Non-Isotropic Heisenberg Groups with Multi-Dimensional Center

- $\mathbb{G}$ is a unimodular Lie group with Haar measure $dz\,dt$.
- Its center is $\{(0,0,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m\}$.
- $m^2 \leq n$.

λ-Fourier–Wigner Transforms

Let $\lambda \in \mathbb{R}^{m*}$. Then for all f and g in $L^2(\mathbb{R}^n)$, we define the λ-Fourier–Wigner transform $V_\lambda(f,g)$ to be the function on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$V_\lambda(f,g)(q,p) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{iB_\lambda^t q\cdot x} f\left(x + \frac{p}{2}\right) \overline{g\left(x - \frac{p}{2}\right)} dx$$

for all $q, p \in \mathbb{R}^n$. Thus,

$$V_\lambda(f,g)(q,p) = V(f,g)(B_\lambda^t q, p), \quad q, p \in \mathbb{R}^n.$$

λ-Wigner Transforms

Let $\lambda \in \mathbb{R}^{m*}$. Then for all f and g in $L^2(\mathbb{R}^n)$, the λ-Wigner transform $W_\lambda(f, g)$ is the function on $\mathbb{R}^n \times \mathbb{R}^n$ defined by

$$W_\lambda(f, g) = V_\lambda(f, g)^\wedge.$$

By a simple calculation,

$$W_\lambda(f, g)(x, \xi) = \frac{1}{\det B_\lambda} W(f, g)(B_\lambda^{-1} x, \xi), \quad x, \xi \in \mathbb{R}^n.$$

λ-Weyl Transforms

Let $\lambda \in \mathbb{R}^{m*}$. Let $\sigma \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then we define the λ-Weyl transform of $f \in L^2(\mathbb{R}^n)$ by

$$(W_\sigma^\lambda f, g)_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(x, \xi) W_\lambda(f, g)(x, \xi)\, dx\, d\xi$$

for all $g \in L^2(\mathbb{R}^n)$. So,

$$(W_\sigma^\lambda f, g)_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\sigma}(q, p) V^\lambda(f, g)(q, p)\, dq\, dp$$

for all $g \in L^2(\mathbb{R}^n)$.

λ-Weyl Transforms

Theorem Let $\sigma \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then

$$W_\sigma^\lambda = W_{\sigma_\lambda},$$

where

$$\sigma_\lambda(x,\xi) = \sigma(B_\lambda x, \xi), \quad x, \xi \in \mathbb{R}^n.$$

Left-Invariant Vector Fields

A basis for left-invariant vector fields on $\mathbb{G}$ is given by $\{X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n, T_k\}$, where

$$(X_j f)(x,y,t) = \frac{\partial f}{\partial x_j}(x,y,t) + \frac{1}{2}\sum_{k=1}^{m}(B_k y, e_j)\frac{\partial f}{\partial t_k}(x,y,t),$$

$$(Y_j f)(x,y,t) = \frac{\partial f}{\partial y_j}(x,y,t) - \frac{1}{2}\sum_{k=1}^{m}(x, B_{kj})\frac{\partial f}{\partial t_k}(x,y,t),$$

$$(T_k f)(x,y,t) = \frac{\partial f}{\partial t_k}(x,y,t).$$

The Sub-Laplacian

The sub-Laplacian $\mathcal{L}$ on $\mathbb{G}$ is given by

$$\mathcal{L} = -\sum_{j=1}^{n} (X_j^2 + Y_j^2).$$

Explicitly,

$$\begin{aligned}\mathcal{L} &= -\Delta_x - \Delta_y - \frac{1}{4}(|x|^2 + |y|^2)\Delta_t \\ &\quad + \sum_{j=1}^{n}\sum_{k=1}^{n}\left[-(B_k y, e_j)\frac{\partial}{\partial x_j} + (x, B_k e_j)\frac{\partial}{\partial y_j}\right]\frac{\partial}{\partial t_k}.\end{aligned}$$

Twisted Laplacians

Taking the inverse Fourier transform of $\mathcal{L}$ with respect to t, we get for $\lambda \in \mathbb{R}^{m*}$,

$$\begin{aligned} L^\lambda &= -\Delta_x - \Delta_y + \frac{1}{4}(|x|^2 + |y|^2)|\lambda|^2 \\ &\quad -i\sum_{j=1}^{n}\left[-(B_\lambda y, e_j)\frac{\partial}{\partial x_j} + (x, B_\lambda e_j)\frac{\partial}{\partial y_j}\right]. \end{aligned}$$

Spectral Analysis of Twisted Laplacians

For all $\lambda \in \mathbb{R}^{m*}$ and multi-indices α and β, we define the function $e^{\lambda}_{\alpha\beta}$ by

$$e^{\alpha}_{\alpha\beta}(q,p) = |\lambda|^{n/2} V^{\lambda}(e_\alpha, e_\beta)\left(\frac{q}{\sqrt{|\lambda|}}, \sqrt{|\lambda|}p\right)$$

for all $q, p \in \mathbb{R}^n$. Then

$$L^{\lambda} e^{\lambda}_{\alpha\beta} = |\lambda|^n (2|\beta| + n) e^{\lambda}_{\alpha,\beta}.$$

The Heat Equation for the Twisted Laplacian

We consider for simplicity the initial value problem for the heat equation generated by the twisted Laplacian. Let $\lambda \in \mathbb{R}^{m*}$. Given

$$\frac{\partial u}{\partial \tau}(z,\tau)(z,\tau) + (L^\lambda u)(z,\tau) = 0, \quad x, y \in \mathbb{R}^n, \tau > 0,$$

with $u(\cdot,\cdot,0) = f$, where f is a given function in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$.

The Heat Kernel

Theorem Let $\lambda \in \mathbb{R}^{m*}$. Then for all $f \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ and $\tau > 0$,

$$(e^{-\tau L^\lambda} f)(z) = \int_{\mathbb{C}^n} \kappa_\tau^\lambda(z,w) f(w)\, dw, \quad z \in \mathbb{C}^n,$$

where

$$\kappa_\tau^\lambda(z,w) = (2\pi)^{-n} \frac{|\lambda|^n}{[2\,\sinh\,(|\lambda|^n \tau)]^n} e^{-\frac{1}{4}|\lambda|\,|z-w|^2 \coth(\tau|\lambda|^n)} e^{-\frac{i}{2}\lambda\cdot[z,w]}$$

for all z and w in $\mathbb{C}^n$.

An L^p-L^∞ Estimate

Theorem Let $\lambda \in \mathbb{R}^{m*}$. Then for $\tau > 0$,

$$e^{-\tau L^\lambda} : L^p(\mathbb{R}^n \times \mathbb{R}^n) \to L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$$

is a bounded linear operator for $1 \le p \le \infty$.

Proof

$$|\kappa_\tau^\lambda(z, w)| \le a_\tau, \quad z, w \in \mathbb{C}^n,$$

where

$$a_\tau = (2\pi)^{-n} \frac{|\lambda|^n}{[2\,\sinh(|\lambda|^n \tau)]^n}.$$

Proof

So,

$$\|e^{-\tau L^\lambda} f\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \le a_t \|f\|_{L^1(\mathbb{R}^n \times \mathbb{R}^n)}.$$

On the other hand,

$$\|e^{-\tau L^\lambda} f\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \le \frac{1}{[\cosh(|\lambda|^n \tau]^n} \|f\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)}.$$

$$\|e^{-\tau L^\lambda} f\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)}$$
$$\le \quad (2\pi)^{-np} \frac{|\lambda|^{np}}{[2\, \sinh(|\lambda|^n \tau)]^{np}} \frac{1}{[\cosh(|\lambda|^n \tau)]^{n(1-p)}} \|f\|_{L^p(\mathbb{R}^n \times \mathbb{R}^n)}.$$

The Heat Semigroup

Theorem Let $\lambda \in \mathbb{R}^{m*}$. The $\tau > 0$,

$$e^{-\tau L^\lambda} f = (\pi)^{n/2} |\lambda|^n \sum_\beta e^{-\tau|\lambda|^n(2|\beta|+n)} V^\lambda(W^\lambda_{\widehat{f}} e_\beta, e_\beta).$$

Proof A key step in the proof is the spectral analysis of the twisted Laplacian using special Hermite functions on $\mathbb{R}^n \times \mathbb{R}^n$ and the spectral mapping theorem.

An L^p-L^2 Estimate for Weyl Transforms

Theorem Let $\sigma \in L^p(\mathbb{R}^n \times \mathbb{R}^n)$ with $1 \le p \le 2$. Then for $\lambda \in \mathbb{R}^{m*}$, the Weyl transform $W_{\hat{\sigma}}^{\lambda}$, originally defined on $\mathcal{S}(\mathbb{R}^n$ can be extended to a unique bounded linear operator on $L^2(\mathbb{R}^n)$. Moreover, $\|W_{\hat{\sigma}}^{\lambda}\|_* \le (2\pi)^{-n/2}(2/|\lambda|)^{n(1-(2/p'))}\|\sigma\|_p, where \| \; \|_*$ is the norm in the C^*-algebra of all bounded linear operators on $L^2(\mathbb{R}^n)$.

An L^p-L^2 Estimate for the Heat Semigroup

Theorem Let $\lambda \in$R^{m*}. Then for $\tau > 0$, $e^{-\tau L^\lambda}$ is a bounded linear operator from $L^p(\mathbb{R}^n \times \mathbb{R}^n)$ into $L^2(\mathbb{R}^n \times \mathbb{R}^n)$, $1 \le 2$, and

$$\|e^{-\tau L^\lambda} f\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)} \le 2^{n(1-(2/p'))} \frac{1}{[2 \sinh(|\lambda|^n \tau)]^n} \|f\|_{L^p(\mathbb{R}^n \times \mathbb{R}^n)}.$$

An L^p-L^q Estimate

Interpolating the L^p-L^2 estimate for $1 \leq p \leq 2$ and the L^p-L^∞ estimate for $1 \leq p \leq \infty$, we get

Theorem Let $\lambda \in \mathbb{R}^{m*}$. Then for $\tau > 0$, $e^{-\tau L^\lambda} : L^p(\mathbb{R}^n \times \mathbb{R}^n) \to L^q(\mathbb{R}^n \times \mathbb{R}^n)$ is a bounded linear operator for all p and q with $1 \leq p \leq 2$ and $2 \leq q \leq \infty$.