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Noncommutative E–conference

Bottom of the L^2 spectrum of the Laplacian on locally symmetric spaces

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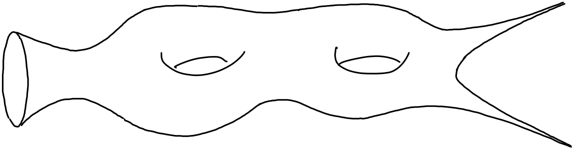
Joint work with Hong–Wei ZHANG [arXiv:2006.06473]

1. Setting

G semisimple Lie group (connected, noncompact, finite center)
K maximal compact subgroup of G
$X = G/K$ Riemannian symmetric space (noncompact type)
Example : hyperbolic spaces
Γ discrete subgroup of G (torsion free)
$Y = \Gamma \backslash G/K$ locally symmetric space (Riemannian manifold)



Cartan decomposition $\sim$ **polar decomposition**:

$$G = K(\exp \overline{\mathfrak{a}^+})K \rightsquigarrow X = K(\exp \overline{\mathfrak{a}^+})K/K$$

Measure : $dg = \omega(r)\, dk_1\, dr\, dk_2$ with $\omega(r) \sim e^{2\langle \rho, r\rangle}$ and $\rho \in \mathfrak{a}^+$

- The positive Weyl chamber $\mathfrak{a}^+$ is a cone in the Euclidean space $\mathfrak{a}$
- The rank of G/K is the dimension ℓ of $\mathfrak{a}$

Laplacians $\begin{cases} \Delta_X & \text{on } X = G/K \\ \Delta_Y & \text{on } Y = \Gamma\backslash G/K \end{cases}$

- The bottom of the L^2 spectrum of $-\Delta_X$ is equal to $|\rho|^2$
- Bottom of the L^2 spectrum of $-\Delta_Y$: $0 \leq \lambda_0 \leq |\rho|^2$

Poincaré series :

$$P_s(\Gamma xK, \Gamma yK) = \sum\nolimits_{\gamma\in\Gamma} e^{-s\,d(\gamma xK, yK)} \qquad (s > 0)$$

- $P_s(\Gamma xK, \Gamma yK) < +\infty$ for one (x, y) $\iff$ for all (x, y)

Critical exponent :

$$\delta = \inf\{\, s > 0 \mid P_s(\Gamma eK, \Gamma eK) < +\infty \,\}$$

- Relation with the counting function

$$\delta = \limsup_{R\to+\infty} \frac{\log|\{\gamma\in\Gamma \mid d(\gamma xK, yK) < R\}|}{R} \quad \forall\, xK, yK \in G/K$$

- $0 \leq \delta \leq 2|\rho|$

2. Rank $\ell = 1$

Hyperbolic spaces

X	$H^n(\mathbb{R})$	$H^n(\mathbb{C})$	$H^n(\mathbb{H})$	$H^2(\mathbb{O})$
G	$SO_0(n,1)$	$SU(n,1)$	$Sp(n,1)$	$F_{4(-20)}$
K	$SO(n)$	$S[U(n)\times U(1)]$	$Sp(n)$	$SO(9)$
d	n	$2n$	$4n$	16
ρ	$\frac{n-1}{2}$	n	$2n+1$	11

Theorem 1 [Elstrodt, Patterson, Sullivan, Corlette]

$$\lambda_0 = \begin{cases} \rho^2 & \text{if } 0 \le \delta \le \rho \\ \rho^2 - (\delta-\rho)^2 & \text{if } \rho \le \delta \le 2\rho \end{cases}$$

Proof : The Green function provides a link between λ_0 and δ

- Green function on $X = G/K$

$$(-\Delta - \rho^2 + \sigma^2)^{-1} f(x) = \int_X g_\sigma^X(x,y)\, f(y)\, dy \qquad (\sigma \geq 0)$$

$$g_\sigma^X(x,y) = C_\sigma\, (\cosh r)^{-\rho-\sigma}\, {}_2\mathrm{F}_1\big(\tfrac{\rho+\sigma}{2}, \tfrac{\rho+\sigma+1}{2}; \sigma+1; \cosh^{-2} r\big)$$

$$\asymp e^{-(\rho+\sigma)\, r} \text{ for large } r = d(x,y)$$

- Green function on $Y = \Gamma \backslash G/K$

$$g_\sigma^Y(\Gamma x K, \Gamma y K) = \sum\nolimits_{\gamma \in \Gamma} g_\sigma^X(y^{-1}\gamma x) \quad \forall\ \Gamma x K \neq \Gamma y K$$

λ_0 is the supremum of $\rho^2 - \sigma^2$, over all $\sigma \in [0, \rho]$ such that

$$g_\sigma^Y(\Gamma x K, \Gamma y K) < +\infty \quad \forall\ \Gamma x K \neq \Gamma y K$$

- $0 \leq \delta \leq \rho$:

$$\sum\nolimits_{\gamma \in \Gamma} e^{-(\rho+\sigma)\, d(\gamma x K, y K)} < +\infty \quad \forall\ 0 \leq \sigma \leq \rho$$

- $\rho \leq \delta \leq 2\rho$:

$$\sum\nolimits_{\gamma \in \Gamma} e^{-(\rho+\sigma)\, d(\gamma x K, y K)} \begin{cases} < +\infty & \text{if } \sigma > \delta - \rho \\ = +\infty & \text{if } \sigma < \delta - \rho \end{cases}$$

□

3. Higher rank

Theorem 2 [Leuzinger, Weber]

$$\begin{cases} \lambda_0 = |\rho|^2 & \text{if } 0 \le \delta \le \rho_{\min} \\ |\rho|^2 - (\delta - \rho_{\min})^2 \le \lambda_0 \le |\rho|^2 & \text{if } \rho_{\min} \le \delta \le |\rho| \\ \max\{0, |\rho|^2 - (\delta - \rho_{\min})^2\} \le \lambda_0 \le |\rho|^2 - (\delta - |\rho|)^2 & \text{if } |\rho| \le \delta \le 2|\rho| \end{cases}$$

where $\rho_{\min} = \inf_{x \in \mathfrak{a}^+, |x|=1} \langle \rho, x \rangle \le |\rho|$

Green function estimate [A–Ji]

$$g_\sigma^X(xK, yK) \asymp \begin{cases} \prod_{\alpha \in R_{\text{red}}^+} (1 + \langle \alpha, r \rangle)\, e^{-\langle \rho, r \rangle - \sigma |r|} & \text{for } r = (y^{-1}x)^+ \text{ large} \\ |r|^{-(d-2)}, \text{ resp. } \log \frac{1}{|r|} & \text{for } r = (y^{-1}x)^+ \text{ small} \end{cases}$$

First improvement

Modified Poincaré series

$$P'_s(\Gamma xK,\Gamma yK)=\sum\nolimits_{\gamma\in\Gamma} e^{-s\,d'(\gamma xK,yK)} \qquad (s>0)$$

where $d'(xK,yK)=\langle\frac{\rho}{|\rho|},(y^{-1}x)^+\rangle$ and $(y^{-1}x)^+$ denotes the $\overline{\mathfrak{a}^+}$–component of $y^{-1}x$ in the Cartan decomposition $G=K(\exp\overline{\mathfrak{a}^+})K$

Modified critical exponent

$$\delta'=\inf\{\,s>0\mid P'_s(\Gamma eK,\Gamma eK)<+\infty\}$$

- $\frac{\rho_{\min}}{|\rho|}\,d\le d'\le d$ and $\delta\le\delta'\le 2|\rho|$
- $\delta'=\limsup_{R\to+\infty}\frac{\log|\{\gamma\in\Gamma\,|\,d'(\gamma xK,yK)<R\}|}{R} \quad \forall\, xK,yK\in G/K$

Theorem 3

$$\begin{cases}\lambda_0=|\rho|^2 & \text{if } 0\le\delta'\le|\rho|\\ |\rho|^2-(\delta'-|\rho|)^2\le\lambda_0\le|\rho|^2 & \text{if } \delta\le|\rho|\le\delta'\\ |\rho|^2-(\delta'-|\rho|)^2\le\lambda_0\le|\rho|^2-(\delta-|\rho|)^2 & \text{if } |\rho|\le\delta\le\delta'\le 2|\rho|\end{cases}$$

Second improvement

Modified Poincaré series

$$P''_s(\Gamma xK, \Gamma yK) = \sum_{\gamma\in\Gamma} e^{-d_s(\gamma xK, yK)} \qquad (s>0)$$

where

$$\begin{aligned} d_s(x,y) &= \min\{s,|\rho|\}\, d'(x,y) + \max\{s-|\rho|,0\}\, d(x,y) \\ &= \begin{cases} s\,d'(x,y) & \text{if } 0<s\le|\rho| \\ s\,d(x,y) - |\rho|\{d(x,y)-d'(x,y)\} & \text{if } s\ge|\rho| \end{cases} \end{aligned}$$

Modified critical exponent

$$\delta'' = \inf\{\, s>0 \mid P''_s(\Gamma eK, \Gamma eK) < +\infty \,\}$$

- $s\,d' \le d_s \le s\,d \quad \forall\, s\ge 0$
- $0 \le \delta \le \delta'' \le \delta' \le 2|\rho|$

Theorem 4

$$\lambda_0 = \begin{cases} |\rho|^2 & \text{if } 0\le\delta''\le|\rho| \\ |\rho|^2 - (\delta''-|\rho|)^2 & \text{if } |\rho|\le\delta''\le 2|\rho| \end{cases}$$

4. Lattice case

- If Γ is a lattice (i.e., $Y = \Gamma \backslash G/K$ has finite volume), then $\lambda_0 = 0$ and $\delta = \delta' = \delta'' = 2|\rho|$
- Conversely, if G has Kazhdan's property (T), then the following conditions are equivalent :
  - Γ is a lattice,
  - $\lambda_0 = 0$,
  - $\delta = 2|\rho|$,
  - $\delta'' = 2|\rho|$.
- Question : What about the condition $\delta' = 2|\rho|$?

Thank you for your attention