Maximal estimate for the Kramers-Fokker-Planck operator with electromagnetic field (after Helffer-Nier, Karaki, Helffer-Karaki).

August 2020

Bernard Helffer,
Université de Nantes.

Abstract

In continuation of a old work by Helffer-Nier (2009) and a recent work by the second author on the torus $\mathbb{T}^d$ ($d = 2, 3$), we consider the Kramers-Fokker-Planck operator (KFP) with an external electromagnetic field on $\mathbb{R}^d$. We show a maximal type estimate on this operator using a nilpotent approach for vector field polynomial operators in relation with induced representations of a nilpotent graded Lie algebra. This estimate leads to an optimal characterization of the domain of the closure of the (KFP)-operator.

Introduction

The Fokker-Planck equation was introduced by Fokker and Planck at the beginning of the twentieth century, to describe the evolution of the density of particles under Brownian motion. In recent years, global hypoelliptic estimates have led to new results motivated by applications to the kinetic theory of gases. In this direction many authors have shown maximal estimates to deduce the compactness of the resolvent of the Fokker-Planck operator and to have resolvent estimates in order to address the issue of return to the equilibrium.

In this talk, we present the study of the model case of the operator of Fokker-Planck with an external magnetic field B_e, and an electric potential and we establish a maximal-type estimate for this model, giving a characterization of the domain of its closed extension (this extends previous works by Hérau-Nier (2008), Helffer-Nier (2009) and Karaki (PHD thesis 2019 and Journal of Spectral Theory (2020)).

Statement of the result

For $d = 2$ or 3, we consider the Kramers-Fokker-Planck operator K with an external electromagnetic field B_e defined on $\mathbb{R}^d$ with value in $\mathbb{R}^{d(d-1)/2}$ and an electric real valued potentiel V defined on $\mathbb{R}^d$:

$$K = v \cdot \nabla_x - \nabla_x V \cdot \nabla_v - (v \wedge B_e) \cdot \nabla_v - \Delta_v + v^2/4 - d/2, \tag{1}$$

where $v \in \mathbb{R}^d$ represents the velocity, $x \in \mathbb{R}^d$ represents the space variable. In the previous definition of our operator, $(v \wedge B_e) \cdot \nabla_v$ means:

$$(v \wedge B_e) \cdot \nabla_v = \begin{cases} b(x)\,(v_1 \partial_{v_2} - v_2 \partial_{v_1}) & \text{if } d = 2 \\ b_1(x)(v_2 \partial_{v_3} - v_3 \partial_{v_2}) + b_2(x)(v_3 \partial_{v_1} - v_1 \partial_{v_3}) & \\ +b_3(x)(v_1 \partial_{v_2} - v_2 \partial_{v_1}) & \text{if } d = 3. \end{cases}$$

The operator K is considered as an unbounded operator on $L^2(\mathbb{R}^d \times \mathbb{R}^d)$ with domain

$$D(K) = C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)\,.$$

We denote by:

- $K_{\min}$ the minimal extension of K where $D(K_{min})$ is the closure of $D(K)$ with respect to the graph norm
- $K_{\max}$ is the maximal extension of K with domain

$$D(K_{\max}) = \{u \in L^2(\mathbb{R}^d \times \mathbb{R}^d)\,/\,Ku \in L^2(\mathbb{R}^d \times \mathbb{R}^d)\}.$$

We use the notation $\mathbf{K}$ for the operator K_{min}.

Maximal accretivity

The existence of a strongly continuous semi-group associated to operator **K** is shown in [ZK1] (Karaki 2019) when the magnetic field is regular. We improve this result by considering a much lower regularity. In order to obtain the maximal accretivity, we are led to substitute the hypoellipticity argument by a regularity argument for the operators with coefficients in L^∞_{loc} introduced in [ZK2] (Karaki 2020), which will be combined with more classical results of Rothschild-Stein (1979) for the operators introduced by Hörmander in 1967:

$$\sum_j X_j^2 + X_0\,.$$

Theorem A

If $B_e \in L^\infty_{loc}(\mathbb{R}^d, \mathbb{R}^{d(d-1)/2})$ and $V \in W^{1,\infty}_{loc}(\mathbb{R}^d)$, then **K** is maximally accretive.

This implies that the domain of the operator $\mathbf{K}$ has the following property:

$$D(\mathbf{K}) = D(K_{\max})\,. \tag{2}$$

Characterization of the domain

We are now interested in specifying the domain of the operator **K**. For this goal, we prove a maximal estimate for **K**, using techniques developed initially by Helffer-Nourrigat for the study of hypoellipticity of invariant operators on graded nilpotent groups.

Notation

- $B_v^2(\mathbb{R}^d) := \{u \,|\, \forall(\alpha,\beta) \in \mathbb{N}^{2d},\ |\alpha| + |\beta| \leq 2\,,\ v^\alpha\, \partial_v^\beta\, u \in L^2\}$
- $\tilde{B}^2(\mathbb{R}^d \times \mathbb{R}^d) := L_x^2 \widehat{\otimes} B_v^2$

We can now state the main theorem of this talk:

Theorem B

Let $d = 2$ or 3. Assume that $B_e \in C^1(\mathbb{R}^d, \mathbb{R}^{d(d-1)/2}) \cap L^\infty$ and $\exists C >$, $\exists \rho_0 > \frac{1}{3}$ and $\exists \gamma_0 < \frac{1}{3}$ s. t.

$$|\nabla_x B_e(x)| \leq C < \nabla V(x) >^{\gamma_0}\,, \tag{3}$$

$$|D_x^\alpha V(x)| \leq C < \nabla V(x) >^{1-\rho_0}\,, \forall \alpha \text{ s.t. } |\alpha| = 2\,, \tag{4}$$

Then $\exists C_1 > 0$ s. t. $\forall u \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$,

$$\begin{aligned} &||\,|\nabla V(x)|^{\frac{2}{3}}\, u\,|| + \|(v \cdot \nabla_x - \nabla_x V \cdot \nabla_v - (v \wedge B_e) \cdot \nabla_v) u\| + \|u\|_{\tilde{B}^2} \\ &\qquad \leq C_1(\, \|Ku\| + \|u\|\,). \end{aligned} \tag{5}$$

Using the density of $C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ in $D(\mathbf{K})$, we obtain:

Corollary B1

$$D(\mathbf{K}) = \{u \in \tilde{B}^2 \mid (v \cdot \nabla_x - \nabla V(x) \cdot \nabla_v - (v \wedge B_e) \cdot \nabla_v)\, u \in L^2 \text{ and } |\nabla V|^{\frac{2}{3}}\, u \in L^2\}.$$

Note that the domain is independent of B_e. Hence the compact resolvent property is independent of B_e. In particular, If in addition $|\nabla V(x)|$ tends to $+\infty$ as $|x| \to +\infty$, then $\mathbf{K}$ has compact resolvent.

Notice also that if $\mathbf{K}$ has compact resolvent then the Witten-Laplacian $-\Delta + \frac{1}{4}|\nabla V|^2 - \frac{1}{2}\Delta V$ has compact resolvent (see Helffer-Nier for the case $B = 0$).

The proofs will combine the previous works of Helffer-Nier (2005) (in the case $B_e = 0$) and [ZK] (in the case $V = 0$) with mainly two differences:

- $\mathbb{T}^d$ is replaced by $\mathbb{R}^d$
- The reference operator in the enveloping algebra of the nilpotent group is different.

Proof of Theorem B: General strategy

The proof consists in constructing a graded and stratified algebra $\mathcal{G}$ of type 2, and, at any point $x \in \mathbb{R}^d$, an homogeneous element $\mathcal{F}_x$ in the enveloping algebra $\mathcal{U}_2(\mathcal{G})$ which satisfies the Rockland condition. We recall that Helffer-Nourrigat's proof is based on maximal estimates which not only hold for the operator but also (and uniformly) for $\pi(\mathcal{F}_x)$ where π is any induced representation of the Lie Algebra.

It remains to find π_x s. t. $\pi_x(\mathcal{F}_x) = \mathcal{K}_x + 1$ is a good approximation of $\mathbf{K}$ in suitable balls and to patch together the estimates through a partition of unity as used by L. Hörmander in his Weyl calculus. This strategy was used in Helffer-Nier in the case $B = 0$ and by Karaki in the case of the torus $\mathbb{T}^d$ with $V = 0$. Actually, we first define $\mathcal{K}_x$ and then look for the Lie Algebra, the operator and the induced representation.

Maximal estimate

For simplification we assume $d = 2$ The dependence on x will actually appear through the two parameters $(b, w) \in \mathbb{R} \times \mathbb{R}^2$ where

$$b = B_e(x), w = \nabla V(x)\,.$$

We then consider the following model:

$$\mathcal{K}_x := K_{w,b} = v \cdot \nabla_x - w.\nabla_v + b(v_1\partial_{v_2} - v_2\partial_{v_1}) - \Delta_v + v^2/4\,. \tag{6}$$

We give uniform estimates with respect to the parameters b, w.

Proposition ME

For any compact interval I, there exists a constant C s. t. for any $b \in I$, any $w \in \mathbb{R}^2$, any $f \in \mathcal{S}(\mathbb{R}^4)$ we have the maximal estimate

$$\begin{aligned} |w|^{\frac{4}{3}}||f||^2 + |w|^{\frac{2}{3}}||\nabla_v f||^2 + |w|^{\frac{2}{3}}||vf||^2 + \textstyle\sum_{|\alpha|+|\beta|\leq 2} ||v^\alpha \partial_v^\beta f||^2 \\ \leq C\left(||f||^2 + ||K_{w,b} f||^2\right)\,. \end{aligned} \tag{7}$$

Proof of Proposition ME. Step 1

Following Helffer-Nier, by the x partial Fourier transform, we get

$$\hat{K}_{w,b,\xi} = iv \cdot \xi - w.\nabla_v + b(v_1\partial_{v_2} - v_2\partial_{v_1}) - \Delta_v + v^2/4 \qquad (8)$$

acting on $L^2(\mathbb{R}^2_v)$.
Its symbol is

$$\sigma(v,\eta) = i\xi \cdot v - iw \cdot \eta - ib(v_1\eta_2 - v_2\eta_1) + \eta^2 + v^2/4 - 1.$$

After conjugation by a metaplectic transfomation associated with a symplectic transformation on $T^*\mathbb{R}^2$, we obtain with the new parameters b', ρ, we get the operator

$$\check{K}_{\rho,b'} = i\, v \cdot \rho + b'_1(v_1\partial_{v_2} - v_2\partial_{v_1}) + ib'_2(v_1 v_2 - \partial_{v_1}\partial_{v_2}) - \Delta_v + v^2/4\,, \qquad (9)$$

with $|w|^2 + \xi^2 = |\rho|^2$ and $|b'| = b$.
It remains to show that for a suitable Lie Algebra $\mathcal{G}$, this is the image by an induced representation π_ρ of an element $\mathcal{F}_{b'}$ satisfying the Rockland condition.

Nilpotent techniques for the analysis of $\check{K}_{\rho,b'}$.

We want to construct

- a graded Lie algebra $\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \mathcal{G}_3$ of type 2 and rank 3
- a subalgebra $\mathcal{H}$, for $b' \in \mathbb{R}^2$ an element $\mathcal{F}_{b'}$ in $\mathcal{U}_2(\mathcal{G})$,
- for $\rho \in \mathbb{R}^2$ a linear form $\ell_\rho \in \mathcal{G}^*$ s. t.
  - $\ell_\rho([\mathcal{H},\mathcal{H}|) = 0$
  - $$\pi_{\ell_\rho,\mathcal{H}}(\mathcal{F}_{b'}) = \check{K}_{\rho,b'}\,.$$

$\check{K}_{\rho,b'}$ can be written as a ρ-independent polynomial of five differential operators

$$X'_{1,1} = \partial_{v_1}\,,\; X''_{1,1} = iv_1\,,\; X'_{2,1} = \partial_{v_2}\,,\; X''_{2,1} = iv_2\,,\; X_{1,2} = i\,v\cdot\rho\,. \tag{10}$$

$$\begin{aligned}\check{K}_{\rho,b'} &= X_{1,2} - \sum_{k=1}^{2}\left((X'_{k,1})^2 + \frac{1}{4}(X''_{k,1})^2\right) \\ &\quad - ib'_1\left(X'_{1,1}X''_{2,1} - X'_{2,1}X''_{1,1}\right) - ib'_2\left(X''_{1,1}X''_{2,1} + X'_{1,1}X'_{2,1}\right)\,.\end{aligned} \tag{11}$$

We now look at the Lie algebra generated by these five operators and their brackets. This leads us to introduce three new elements that verify the following relations :

$$X_{2,2} := [X_{1,1}^{'}, X_{1,1}^{''}] = [X_{2,1}^{'}, X_{2,1}^{''}] = i \,,$$
$$X_{1,3} := [X_{1,2}, X_{1,1}^{'}] = -i\rho_1 \,,\; X_{2,3} := [X_{1,2}, X_{2,1}^{'}] = -i\rho_2 .$$

We also observe that we have the following properties:

$$[X_{1,1}^{'}, X_{2,1}^{'}] = [X_{1,1}^{''}, X_{2,1}^{''}] = 0 \,,$$
$$[X_{j,1}^{'}, X_{k,3}] = [X_{j,1}^{''}, X_{k,3}] = [X_{k,3}, X_{2,2}] = ... = 0 \quad , \forall j, k = 1, 2 \,.$$

We then construct a graded Lie algebra $\mathcal{G}$ verifying the same commutator relations.

- $\mathcal{G}_1$ is generated by $Y'_{1,1}, Y'_{2,1}, Y''_{1,1}$ and $Y''_{2,1}$,
- $\mathcal{G}_2$ is generated by $Y_{1,2}$ and $Y_{2,2}$
- $\mathcal{G}_3$ is generated by $Y_{1,3}$ and $Y_{2,3}$.

$$Y_{2,2} := [Y'_{1,1}, Y''_{1,1}] = [Y'_{2,1}, Y''_{2,1}],$$
$$Y_{1,3} := [Y_{1,2}, Y'_{1,1}]\,,\; Y_{2,3} := [Y_{1,2}, Y'_{2,1}]\,,$$

and

$$[Y'_{1,1}, Y'_{2,1}] = [Y''_{1,1}, Y''_{2,1}] = 0\,,$$
$$[Y'_{j,1}, Y_{k,3}] = [Y''_{j,1}, Y_{k,3}] = [Y_{k,3}, Y_{2,2}] = \ldots = 0 \quad , \forall j,k = 1,2\,.$$

We have to check that for a given $\rho = (\rho_1, \rho_2)$ the representation π (with the convention that if $\diamond = \emptyset$ there is no exponent) defined on its basis by

$$\pi(Y_{i,j}^{\diamond}) = X_{i,j}^{\diamond} \text{ with } i = 1, 2, j = 1, 2, 3 \text{ and } \diamond \in \{\emptyset, {}', {}''\}. \tag{12}$$

defines an induced representation of $\mathcal{G}$.
We obtain $\pi = \pi_{\ell,\mathcal{H}}$ with

$$\mathcal{H} = \mathrm{Vect}(Y_{1,1}'', Y_{2,1}'', Y_{1,2}, Y_{2,2}, Y_{1,3}, Y_{2,3}), \tag{13}$$

and $\ell_\rho \in \mathcal{G}*$ defined by 0 for the elements of the basis of $\mathcal{G}$ except

$$\ell_\rho(Y_{1,3}) = -\rho_1, \ell_\rho(Y_{2,3}) = -\rho_2, \ell_\rho(Y_{2,2}) = 1\,. \tag{14}$$

We then introduce

$$\mathcal{F}_{b'} = Y_{1,2} - \sum_{k=1}^{2} \left((Y'_{k,1})^2 + \frac{1}{4}(Y''_{k,1})^2 \right) - ib'_1 \left(Y'_{1,1} Y''_{2,1} - Y'_{2,1} Y''_{1,1} \right) - ib'_2 \left(Y''_{1,1} Y''_{2,1} + Y'_{1,1} Y'_{2,1} \right) \tag{15}$$

and get

$$\pi_{\ell_\rho,\mathcal{H}}(\mathcal{F}_{b'}) = \check{K}_{\rho,b'} \,. \tag{16}$$

The verification of the Rockland condition is easy using that $\pi(Y)$ is a formally skew-adjoint operator $\forall Y \in \mathcal{G}$. Therefore, according to the Helffer-Nourrigat theorem the operator $\mathcal{F}_{b'}$ is maximal hypoelliptic and we have a locally uniform (with respect to b') maximal estimate.
We obtain $\forall K_0 \subset \mathbb{R}^2\ \exists C > 0$ s. t. $\forall b' \in K_0$, $\forall \rho \in \mathbb{R}^2$, $\forall u \in \mathcal{S}(\mathbb{R}^2)$,

$$\|X_{1,2}u\|^2 + \sum_{k=1}^{2} \left(\|(X'_{k,1})^2 u\|^2 + \|(X''_{k,1})^2 u\|^2\right) + \sum_{k,\ell=1}^{2} \|X'_{k,1}X''_{\ell,1}u\|^2 \leq C\ \left(\|\check{K}_{\rho,b'}u\|^2 + \|u\|^2\right)\ . \tag{17}$$

In particular, we have

$$\|(v\cdot\rho)\,u\|^2 + \sum_{|\alpha|+|\beta|\leq 2} ||v^\alpha \partial_v^\beta u||^2 \leq C\,\left(\|\check{K}_{\rho,b'}u\|^2 + \|u\|^2\right)\,. \quad (18)$$

On the other hand $\exists \check{C}$, s. t. , $\forall \rho \in \mathbb{R}^2$ and $\forall u \in \mathcal{S}(\mathbb{R}^2)$, we have

$$|\rho|^{\frac{4}{3}}||u||^2 \leq \check{C}(||(-\Delta + v\cdot\rho)u)||^2)\,. \quad (19)$$

End of the proof of Proposition ME

Coming back to the initial coordinates, we get for a new constant $\tilde{C} > 0$,

$$||f||^2 + ||\hat{K}_{w,b,\xi} f||^2 \geq \frac{1}{\tilde{C}} \left(|w|^{\frac{2}{3}} ||vf||^2 + |w|^{\frac{2}{3}} ||\nabla_v f||^2 + \sum_{|\alpha|+|\beta|\leq 2} ||v^\alpha \partial_v^\beta f||^2 \right) . \quad (20)$$

Considering the inverse Fourier transform, this achieves the proof of the maximal estimate.

Proof of Theorem B–Partition of unity.

One can now introduce $\forall \delta \in (0, \delta_0])$ a δ-dependent partition of unity ϕ_j in the x variable corresponding to a covering by balls $B(x_j, r(x_j, \delta))$ (with the property of uniform finite intersection N_δ). The support of each ϕ_j is contained in $B(x_j, \delta r(x_j))$ and we have

$$\sum_j \phi_j^2(x) = 1\,, \tag{21}$$

and

$$|\nabla_x \phi_j(x)| \leq C_\delta < \nabla_x V(x_j) >^s \leq \hat{C}_\delta < \nabla_x V(x) >^s \quad . \tag{22}$$

This implies (using the finite intersection property)

$$\sum_j |\nabla_x \phi_j(x)|^2 \leq \check{C}_\delta < \nabla_x V(x) >^{2s} \quad . \tag{23}$$

The core of the proof

The proof is inspired by the chapter 9 of [HeNi] with modifications. We start, for $u \in C_0^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$, from

$$\begin{aligned} ||Ku||^2 &= \sum_j ||\phi_j K u||^2 \\ &= \sum_j ||K\phi_j u||^2 - \sum_j ||[K, \phi_j] u||^2 \\ &= \sum_j ||K\phi_j u||^2 - \sum_j ||(X_0 \phi_j) u||^2 \\ &= \sum_j ||K\phi_j u||^2 - \sum_j ||(\nabla \phi_j)(x) v u||^2 \\ &\geq \sum_j ||K\phi_j u||^2 - \check{C}_\delta || < \nabla V(x) >^s \, v u||^2 \, . \end{aligned}$$

For the analysis of $||K\phi_j u||^2$ we write, with $w_j = \nabla_x V(x_j)$ and $b_j = B_e(x_j)$,

$$K = K - K_{w_j,b_j} + K_{w_j,b_j}\,.$$

We verify that, by using the construction of the partition of unity and the assumptions of Theorem B the errors are controlled by the main term.

We note that by using the inequality (20), we have

$$||K_{w_j,b_j}\phi_j u||^2 \geq \frac{1}{C}|| \, |\nabla V(x)|^{\frac{2}{3}}\phi_j u||^2 + \frac{1}{C}|| \, |\nabla V(x)|^{\frac{1}{3}}\phi_j \nabla_v u||^2 + \frac{1}{C}|| \, |\nabla V(x)|^{\frac{1}{3}}\phi_j v u||^2 + \frac{1}{C}\sum_{|\alpha|+|\beta|\leq 2} ||v^\alpha \partial_v^\beta \phi_j u||^2 \, .$$

Finally, we observe that

$$||K\phi_j u||^2 \geq \frac{1}{2}||K_{w_j,b_j}\phi_j u||^2 - ||(K - K_{w_j,b_j})\phi_j u||^2 \, .$$

We now choose

$$\max(\frac{2}{3} - \rho_0, \gamma_0) < s < \frac{1}{3} \, . \tag{24}$$

We can achieve the proof by choosing δ small enough and obtain the existence of a constant $C > 0$ s. t.

$$\begin{array}{l} ||u||^2 + ||Ku||^2 \\ \quad \geq \frac{1}{C} \Big(|| \, |\nabla_x V(x)|^{\frac{2}{3}} \, u \, ||^2 \\ \qquad + || |\nabla_x V(x)|^{\frac{1}{3}} \, \nabla_v u \, ||^2 + |\nabla_x V(x)|^{\frac{1}{3}} \, vu \, ||^2 \\ \qquad + ||(v_1 \partial_{v_2} - v_2 \partial_{v_1}) u||^2 \Big) \\ \qquad\qquad - C(||\nabla_v u||^2 + ||vu||^2 + ||u||^2) \, . \end{array}$$

Combining this inequality with the standard inequality

$$\Re < Ku, u > \geq ||\nabla_v u||^2 + ||vu||^2 - ||u||^2 \, ,$$

this achieves the proof of the theorem.

Thank you for your attention.

Bibliography.

M. Bramanti and L. Brandolini.
L^p estimates for uniformly hypoelliptic operators with discontinuous coefficients on homogeneous groups.
Rend. Sem. Mat. Univ. Politec. Torino, 58(4):389–433, 2000.

K.J. Engel and R. Nagel.
One-parameter semigroups for linear evolution equations.
In *Semigroup Forum*, volume 63, pages 278–280. Springer, 2001.

A.D. Fokker.
Ann. d. Physik, 43:812, 1914.

B. Helffer and F. Nier.
Hypoelliptic estimates and spectral theory for Fokker-Planck operators and Witten Laplacians.
Springer, 2005. Lecture Notes in Mathematics.

B Helffer and J Nourrigat.

Hypoellipticité pour des groupes nilpotents de rang de nilpotence 3.
Communications in Partial Differential Equations, 3(8):643–743, 1978.

B. Helffer and J. Nourrigat.
Caractérisation des opérateurs hypoelliptiques homogènes.
Communications in Partial Differential Equations, 4(8):899–958, 1979.

B. Helffer and J.Nourrigat.
Hypoellipticité maximale pour des opérateurs polynômes de champs de vecteurs.
Birkhäuser *Progress in Mathematics*, 58, 1985.

F. Hérau and F. Nier.
Isotropic hypoellipticity and trend to equilibrium for the Fokker-Planck equation with a high-degree potential.
Archive for Rational Mechanics and Analysis, 171(2):151–218, 2004.

L. Hörmander.
Hypoelliptic second order differential equations.
Acta Mathematica, 119(1):147–171, 1967.

Z. Karaki.
Trend to the equilibrium for the Fokker-Planck system with an external magnetic field.
Kinetic & Related Models, 13(2):309, 2020.

Z. Karaki.
Estimations maximales pour l'opérateur de Fokker-Planck avec champ magnétique.
hal-01975164, 2019 C'est l'article soumis à JST ?

L. P. Rothschild and E.M. Stein.
Hypoelliptic differential operators and nilpotent groups.
Acta Mathematica, 137(1):247–320, 1976.

A. Pascucci.
Hölder regularity for a Kolmogorov equation.

Transactions of the American Mathematical Society, pages 901–924, 2003.