Some results on the p–generalized complete elliptic integrals and associated properties

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Abstract

We introduce a general family of p–generalized complete elliptic integrals $\mathcal{K}_{a,p}(r)$ and $\mathcal{E}_{a,p}(r)$ of the first and the second kind by making use of p–extended Gauss's hypergeometric function, for which the usual properties and representations are extended in a simple manner. Log–convexity property and Turán–type inequalities are proved for these p–generalized elliptic integrals. In addition, we deduce several special values and provides connections with the Meijer G-function as new representations for special parameter, functional bounds, Mellin transforms, certain infinite series representations containing Laguerre polynomials and establish numerous differentiation formulas.

Introduction

In 1994, Chaudhry *et al.* [7, p. 20, Equation (1.7)] introduced the p–extended Beta function as:

$$B(x,\, y\,;\, p) := \int_0^1 t^{x-1}\,(1-t)^{y-1}\,\exp\left(-\frac{p}{t(1-t)}\right)\,dt \qquad (1)$$

$(\Re(p) > 0;$ For $p = 0,\ \Re(x) > 0,\ \Re(y) > 0),$

and studied various properties and obtained certain connections with Macdonald, error and Whittaker functions. Further more interesting properties and various connections with higher transcendental functions are investigated by Miller [13]. Here we call extended Beta function as p–extended Beta function.

In 2004, Chaudhry *et al.* [8] introduced the p–Gauss's hypergeometric and p–Kummer's confluent hypergeometric functions by making use of the p–extended Beta function $B(x,\, y\,;\, p)$ as:

$$F_p\,(a,\; b;\; c;\; z) := \sum_{n=0}^{\infty} (a)_n\,\frac{B(b+n,\, c-b\,;\, p)}{B(b,\, c-b)}\,\frac{z^n}{n!} \qquad (2)$$

$$\big(p \geqq 0,\ |z| < 1\,;\, \Re(c) > \Re(b) > 0\big)$$

and

$$\Phi_p\,(b;\; c;\; z) := \sum_{n=0}^{\infty} \frac{B(b+n,\, c-b\,;\, p)}{B(b,\, c-b)}\,\frac{z^n}{n!} \qquad (3)$$

$$\big(p \geqq 0\,;\, \Re(c) > \Re(b) > 0\big).$$

Main Objectives

For $\Re(p) > 0$, $r \in (0,1)$, $a \in (0,1)$ and $r' = \sqrt{1-r^2}$, we introduce p–generalized elliptic integrals of the first and second kind as:

$$\begin{cases} \mathcal{K}_{a,p} = \mathcal{K}_{a,p}(r) = \frac{\pi}{2}\, F_p\left(a, 1-a; 1; r^2\right), \\ \mathcal{K}'_{a,p} = \mathcal{K}'_{a,p}(r) = \mathcal{K}_{a,p}(r') = \mathcal{K}_{a,p}(\sqrt{1-r^2}) \end{cases} \qquad (4)$$

and

$$\begin{cases} \mathcal{E}_{a,p} = \mathcal{E}_{a,p}(r) = \frac{\pi}{2}\, F_p\left(a-1, 1-a; 1; r^2\right), \\ \mathcal{E}'_{a,p} = \mathcal{E}'_{a,p}(r) = \mathcal{E}_{a,p}(r') = \mathcal{E}_{a,p}(\sqrt{1-r^2}). \end{cases} \qquad (5)$$

Some Properties of p–generalized elliptic integrals

Integral Representations

Theorem 1. For $\Re(p) > 0$, $r \in (0,1)$ and $a \in (0,1)$, the following integral representations for $\mathcal{K}_{a,p}(r)$ and $\mathcal{E}_{a,p}(r)$ in (4) and (5) holds true:

$$\begin{aligned} \mathcal{K}_{a,p}(r) &= \frac{\pi}{2\,\Gamma(a)\,\Gamma(1-a)} \int_0^1 t^{-a}\,(1-t)^{a-1}\,(1-r^2t)^{-a}\,\mathrm{e}^{-\frac{p}{t(1-t)}}\,\mathrm{d}t \\ &= \frac{\sin(\pi a)}{2} \int_0^1 t^{-a}\,(1-t)^{a-1}\,(1-r^2t)^{-a}\,\mathrm{e}^{-\frac{p}{t(1-t)}}\,\mathrm{d}t \end{aligned} \qquad (6)$$

and

$$\begin{aligned} \mathcal{E}_{a,p}(r) &= \frac{\pi}{2\,\Gamma(a)\,\Gamma(1-a)} \int_0^1 t^{-a}\,(1-t)^{a-1}\,(1-r^2t)^{1-a}\,\mathrm{e}^{-\frac{p}{t(1-t)}}\,\mathrm{d}t \\ &= \frac{\sin(\pi a)}{2} \int_0^1 t^{-a}\,(1-t)^{a-1}\,(1-r^2t)^{1-a}\,\mathrm{e}^{-\frac{p}{t(1-t)}}\,\mathrm{d}t. \end{aligned} \qquad (7)$$

Theorem 2. Let $\Re(p) > 0$. Then the following assertions are true:

1. The function $a \mapsto \mathbb{K}_{a,p}(r)$ is log–convex on $(0,1)$ for all $r \in (0,1)$.
2. The function $a \mapsto \mathbb{E}_{a,p}(r)$ is log–convex on $(0,1)$ for all $r \in (0,1)$.

Moreover, for the same parameter range there holds the Turán inequalities

$$\mathbb{K}^2_{\frac{a_1+a_2}{2},p}(r) - \mathbb{K}_{a_1,p}(r)\,.\,\mathbb{K}_{a_2,p}(r) \le 0 \qquad a_1, a_2 \in (0,1), \qquad (8)$$

$$\mathbb{E}^2_{\frac{a_1+a_2}{2},p}(r) - \mathbb{E}_{a_1,p}(r)\,.\,\mathbb{E}_{a_2,p}(r) \le 0 \qquad a_1, a_2 \in (0,1). \qquad (9)$$

Mellin transform

Theorem 3. For $\Re(s) > 0$, $r \in (0,1)$ and $a \in (0,1)$, the following Mellin transform formula for $\mathcal{K}_{a,p}(r)$ and $\mathcal{E}_{a,p}(r)$ in (4) and (5) holds true:

$$\mathcal{M}\left\{\mathcal{K}_{a,p}(r) : p \to s\right\} = \frac{\pi}{2}\,\Im\;{}_2F_1\left(a, 1-a+s; 2s+1; r^2\right), \qquad (10)$$

$$\mathcal{M}\left\{\mathcal{E}_{a,p}(r) : p \to s\right\} = \frac{\pi}{2}\,\Im\;{}_2F_1\left(a-1, 1-a+s; 2s+1; r^2\right) \qquad (11)$$

where

$$\Im = \frac{\Gamma(s)\mathrm{B}(1-a+s,\, a+s)}{\mathrm{B}(1-a,\, a)}$$

Special values representation

Theorem 4. For $\Re(p) > 0$ and $a \in (0,1)$, the following relations holds true:

$$\begin{aligned} \mathcal{K}_{a,p}(0) = \mathcal{K}'_{a,p}(1) &= \frac{\pi}{2}\frac{\mathrm{B}(1-a,\, a\,;\, p)}{\mathrm{B}(1-a,\, a)} = \frac{\sin(\pi a)}{2}\mathrm{B}(1-a,\, a\,;\, p), \\ \mathcal{K}_{a,p}(1) = \mathcal{K}'_{a,p}(0) &= \frac{\pi}{2}\frac{\mathrm{B}(1-a,\, 0\,;\, p)}{\mathrm{B}(1-a,\, a)} = \frac{\sin(\pi a)}{2}\mathrm{B}(1-a,\, 0\,;\, p), \\ \mathcal{E}_{a,p}(0) = \mathcal{E}'_{a,p}(1) &= \frac{\pi}{2}\frac{\mathrm{B}(1-a,\, a\,;\, p)}{\mathrm{B}(1-a,\, a)} = \frac{\sin(\pi a)}{2}\mathrm{B}(1-a,\, a\,;\, p), \\ \mathcal{E}_{a,p}(1) = \mathcal{E}'_{a,p}(0) &= \frac{\pi}{2}\frac{\mathrm{B}(1-a,\, 1\,;\, p)}{\mathrm{B}(1-a,\, a)} = \frac{\sin(\pi a)}{2}\mathrm{B}(1-a,\, 1\,;\, p). \end{aligned}$$

p–Appell's hypergeometric function representations

Theorem 5. For $\Re(p) > 0$, $r \in (0,1)$ and $a \in (0,1)$, the following p–Appell's hypergeometric function representations holds true:

$$\mathcal{K}_{a,p}(r) = \frac{\sin(\pi a)}{2(1-a)}\,F_1\left(1-a, 1-a, a; 2-a; 1, r^2; p\right), \qquad (12)$$

$$\mathcal{E}_{a,p}(r) = \frac{\sin(\pi a)}{2(1-a)}\,F_1\left(1-a, 1-a, a-1; 2-a; 1, r^2; p\right). \qquad (13)$$

Conclusions

In this present talk, we extend the p–generalized elliptic integrals, which are based upon the definition of the p–extended Gauss's hypergeometric function $F_p(a, b; c; z)$. The p–extensions proposed in this paper will be seen to be extremely useful. Many of the known properties of the elliptic integrals, in particular, Log–convexity property and Turán–type inequalities carry over naturally, consult for instance for recent papers on Turán–type inequalities of special function. Furthermore, Mellin transform formulas are obtained for these p–generalized elliptic integrals and provides connections with the Meijer G-function as new representations for special parameter values of the p–elliptic integrals. We also give some infinite series representations containing the Laguerre polynomials and differentiation formulas.

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