Some results on the $p$–generalized complete elliptic integrals and associated properties

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Introduction

We introduce a general family of $p$–generalized complete elliptic integrals $K_{a,p}(r)$ and $E_{a,p}(r)$ of the first and the second kind by making use of $p$–extended Gauss’s hypergeometric function, for which the usual properties and representations are extended in a simple manner. Log–convexity property and Turán–type inequalities are proved for these $p$–generalized elliptic integrals. In addition, we deduce several special values and provides connections with more interesting and studied various properties and obtained certain connections with Macdonald, error and Whittaker functions. Further more interesting and properties and various connections with higher transcendental functions are investigated by Miller [13]. Here we call extended Beta function as new representations for special parameter values of the $p$–generalized elliptic integrals, Mellin transform formulas are obtained for these geometric function $F_{a,p}(r)$ as new representations for special parameter, functional bounds, and establish numerous differentiation formulas.

Abstract

In 1994, Chaudhry et al. [7, p. 20, Equation (1.7)] introduced the $p$–extended Beta function as:

$$B(x, y; p) = \int_0^1 t^{x-1} (1-t)^{y-1} \exp \left(-\frac{p}{t(1-t)} \right) dt$$

(1)

$(R(p) > 0)$. For $p = 0$, $(R(0) > 0)$, and studied various properties and obtained certain connections with Macdonald, error and Whittaker functions. Further more interesting and properties and various connections with higher transcendental functions are investigated by Miller [13]. Here we call extended Beta function as new representations for special parameter values of the $p$–generalized elliptic integrals.

Some Properties of $p$–generalized elliptic integrals

Integral Representations

Theorem 1. For $|R(p)| > 0$, $r \in (0, 1)$ and $a \in (0, 1)$, the following integral representations for $K_{a,p}(r)$ and $E_{a,p}(r)$ in (4) and (5) holds true:

$$K_{a,p}(r) = \frac{n}{2} \left( 1 - \frac{1}{a^2} \right)^{1/2} \left( 1 - r^2 \right)^{a/2} \int_0^\infty \exp \left(-\frac{p}{t(1-t)} \right) dt$$

(6)

and

$$E_{a,p}(r) = \frac{n}{2} \left( 1 - \frac{1}{a^2} \right)^{1/2} \left( 1 - r^2 \right)^{a/2} \int_0^\infty \exp \left(-\frac{p}{t(1-t)} \right) dt$$

(7)

Theorem 2. Let $|R(p)| > 0$. Then the following assertions are true:
1. The function $a \to K_{a,p}(r)$ is log–convex on $(0, 1)$ for all $r \in (0, 1)$.
2. The function $a \to E_{a,p}(r)$ is log–convex on $(0, 1)$ for all $r \in (0, 1)$.

Moreover, for the same parameter range there holds the Turán inequalities

$$K_{a,p}(r) - K_{a+1,p}(r) \leq a \left( a+1 \right) \quad (8)$$

$$E_{a,p}(r) - E_{a,1,p} \leq a \left( a+1 \right) \quad (9)$$

Mellin transform

Theorem 3. For $R(a) > 0$, $r \in (0, 1)$ and $a \in (0, 1)$, the following Mellin transform formula for $K_{a,p}(r)$ and $E_{a,p}(r)$ in (4) and (5) holds true:

$$M \{ K_{a,p}(r) \} = \frac{1}{a} \left( 1 - a \right) \pi r^2$$

(10)

and

$$M \{ E_{a,p}(r) \} = \frac{1}{a} \left( 1 - a \right) \pi r^2$$

(11)

where

$$D \{ F_{a,p}(r) \} = \frac{\Gamma(1-a-s,a+s)}{\Gamma(1-a,a+s)}$$

Special values representation

Theorem 4. For $|R(p)| > 0$ and $a \in (0, 1)$, the following relations holds true:

$$K_{a,p}(0) = K_{a,0}(0) = \frac{\pi}{2} \left( 1 - a \right)$$

$$E_{a,p}(0) = E_{a,0}(0) = \frac{\pi}{2} (1-a)$$

(12)

(13)

Conclusions

In this present talk, we extend the $p$–generalized elliptic integrals, which are based upon the definition of the $p$–extended Gauss’s hypergeometric function $F_{a,p}(r)$. The $p$–extensions proposed in this paper will be seen to be extremely useful. Many of the known properties of the elliptic integrals, in particular, Log–convexity property and Turán–type inequalities carry over naturally, consult for instance for recent papers on Turán–type inequalities of special function. Furthermore, Mellin transform formulas are obtained for these $p$–generalized elliptic integrals and provides connections with the Meijer G–function as new representations for special parameter values of the $p$–elliptic integrals. We also give some infinite series representations containing the Laguerre polynomials and differentiation formulas.

References


