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Dunkl-Hausdorff operators on $\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})$ and $H\dot{K}^{\beta,p,N}_{\alpha,q}(\mathbb{R})$

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Abstract

In the present paper, we have studied the Dunkl–Hausdorff operators $\mathcal{H}_{\alpha,\varphi}$ on the Dunkl-type homogeneous weighted Herz spaces $\dot{K}^{\beta,p}_{\alpha,q}$ and Dunkl Herz-type Hardy space $H\dot{K}^{\beta,p,N}_{\alpha,q}(\mathbb{R})$. We have determined simple sufficient conditions for these operators to be bounded on these spaces. As applications we provide necessary and sufficient conditions for Dunkl-Cesàro operator and sufficient conditions for Dunkl-Hardy operator to be bounded on the homogeneous weighted Herz space $\dot{K}^{\beta,p}_{\alpha,q}$.

Introduction

Let $\varphi \in L^1(\mathbb{R})$. First of all, we will start by recalling the definition of the Dunkl-Hausdorff operator (see [4, 6, 7, 8, 12])

$$\mathcal{H}_{\alpha,\varphi}f(x) := \int_0^\infty \frac{\varphi(t)}{t^{2\alpha+2}} f(\frac{x}{t})dt \qquad (0.1)$$

When $\alpha = -1/2$, The operator $\mathcal{H}_{\alpha,\varphi}$ is the famous Hausdorff operator

$$\mathcal{H}_{\varphi}f(x) = \int_0^\infty \frac{\varphi(t)}{t} f(\frac{x}{t})dt,$$

from which several well known operators can be deduced for suitable choices of φ, e.g., for $\varphi(t) = \frac{1}{t}\chi_{(1,\infty)}(t)$, the operator $\mathcal{H}_\varphi$ reduces to the standard Hardy averaging operator

$$\mathcal{H}f(x) = \frac{1}{x}\int_0^x f(t)\,dt$$

while for $\varphi(t) = \chi_{[0,1]}(t)$, it reduces to the adjoint of Hardy averaging operator

$$\mathcal{H}^* f(x) = \int_x^\infty \frac{f(t)}{t}\,dt.$$

For more details of its historical development, background and some applications, the reader can see a recent survey article [11] by E.Liflyand which contains the main results on Hausdorff operators in various settings and bibliography up 2013.

1 Boundedness of $\mathcal{H}_{\alpha,\varphi}$ on the homogeneous weighted Herz space $\dot{K}^{\beta,p}_{\alpha,q}$.

Let $\beta \in \mathbb{R}$, $0 < p < +\infty$, and $1 \le q < +\infty$. The homogeneous weighted Herz space $\dot{K}^{\beta,p}_{\alpha,q}$ is the space constituted by all the functions $f \in L^q_\alpha(\mathbb{R})_{\mathbf{loc}}$, such that

$$\|f\|_{\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})} := \Big(\sum_{k=-\infty}^{+\infty} 2^{2(\alpha+1)\beta kp}\|f\chi_k\|^p_{L^q_\alpha(\mathbb{R})}\Big)^{\frac{1}{p}} < +\infty,$$

where χ_k is the characteristic function of the set

$$A_k = \{x \in \mathbb{R}\,;\, 2^{k-1} \le |x| \le 2^k\} \quad \text{for} \quad k \in \mathbb{Z},$$

and $L^q_\alpha(\mathbb{R})_{\mathbf{loc}}$ is the space $L^q_{\mathbf{loc}}(\mathbb{R}, |x|^{2\alpha+1}dx)$.
Note that $\dot{K}^{\beta,0}_{\alpha,q}(\mathbb{R}) = L^q_\alpha(\mathbb{R})$.

The main result of this subsection is the following .

Theorem 1 Let $\alpha \ge \frac{-1}{2}$, $\beta \in \mathbb{R}$, $1 < p < +\infty$, $1 \le q < +\infty$, and φ a measurable function on $\mathbb{R}$ such that

$$C_{q,\alpha,\beta,\varphi} = \int_0^\infty |\varphi(t)|t^{2(\alpha+1)(\beta-1+\frac{1}{q})}dt < \infty.$$

Then, the Dunkl-Hausdorff operator $\mathcal{H}_{\alpha,\varphi}$ is bounded from $\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})$ to itself, i.e.,

$$\|\mathcal{H}_{\alpha,\varphi}f\|_{\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})} \lesssim C_{q,\alpha,\beta,\varphi}\|f\|_{\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})}.$$

If φ is supported in the interval $[0,1]$, then $\mathcal{H}_{\alpha,\varphi}$ reduces to the Dunkl-Cesàro operator $\mathcal{C}_{\alpha,\varphi}$ defined by

$$\mathcal{C}_\varphi f(x) := \int_0^1 \frac{\varphi(t)}{t^{2\alpha+2}} f(\frac{x}{t})dt, \quad x \in \mathbb{R},$$

(see [4, 1]).

Corollary 1.
Let $\alpha \ge \frac{-1}{2}$, $\beta \in \mathbb{R}^\times$, $1 < p < +\infty$, $1 \le q < +\infty$, and φ a non-negative measurable function defined on $[0,1]$.

Then, the generalized Cesàro operator $\mathcal{C}_{\alpha,\varphi}$ is bounded from $\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})$ to itself if and only if

$$C_{q,\alpha,\varphi} = \int_0^1 \varphi(t)t^{2(\alpha+1)(\beta+\frac{1}{q}-1)}dt < \infty.$$

If $\varphi(t) = \frac{\chi_{(1,\infty)}(t)}{t}$, then (0.1) is of the following form:

$$\mathcal{H}_\alpha f(x) = \frac{1}{x^{2\alpha+1}}\int_0^x f(\xi)d\mu_\alpha(\xi).$$

In this case, $\mathcal{H}_{\alpha,\varphi}$ reduces to the Dunkl-Hardy operator for which we deduce this result.

Corollary 2. Let $\alpha \ge \frac{-1}{2}$, $1 < p < +\infty$, $1 \le q < +\infty$, $0 < \beta < 1 - \frac{1}{q}$, and $\varphi(t) = \frac{\chi_{(1,\infty)}(t)}{t}$.

Then, the Dunkl-Hardy operator $\mathcal{H}_{\alpha,\varphi}$ is bounded from $\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})$ to itself and we have

$$\|\mathcal{H}_{\alpha,\varphi}f\|_{\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})} \le \frac{1}{2(\alpha+1)(\beta-1+\frac{1}{q})}\|f\|_{\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})}.$$

2 Boundedness of $\mathcal{H}_{\alpha,\varphi}$ on the Dunkl Herz-type Hardy space $H\dot{K}^{\beta,p,N}_{\alpha,q}(\mathbb{R})$.

Let $\alpha \ge \frac{-1}{2}$, $N \in \mathbb{N}$, $\beta \in \mathbb{R}$, $0 < p < +\infty$, and $1 \le q < +\infty$. The Herz-type Hardy space $H\dot{K}^{\beta,p,N}_{\alpha,q}(\mathbb{R})$ is the space of distributions $f \in \mathcal{S}'(\mathbb{R})$ such that $G_{\alpha,N}(f) \in \dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})$. Moreover, we have

$$\|f\|_{H\dot{K}^{\beta,p,N}_{\alpha,q}(\mathbb{R})} = \|G_{\alpha,N}(f)\|_{\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})}$$

In the sequel, we are interested in the spaces $H\dot{K}^{\beta,p,N}_{\alpha,q}(\mathbb{R})$, when $\beta \ge 1 - 1/q$. Now, we turn to the atomic characterization of the space $H\dot{K}^{\beta,p,N}_{\alpha,q}(\mathbb{R})$.

Let $\alpha \ge \frac{-1}{2}$, $1 \le q \le \infty$, and $\beta \ge 1 - 1/q$. A measurable function a on $\mathbb{R}$ is called a (central) $(\beta;q)$-atom if it satisfies:

1. $suppa \subset [-r, r]$, for a certain $r > 0$,

2. $\|a\|_{q,\alpha} \le r^{-2(\alpha+1)\beta}$,

3. $\int_{\mathbb{R}} a(x)x^k d\mu_\alpha(x) = 0$, $k = 0, 1, \cdots, 2s+1$,

where s is the integer part of $(\alpha+1)(\beta-1+1/q)$.

Theorem 2 Let $\alpha \ge \frac{-1}{2}$, $0 < p < +\infty$, $1 \le q < +\infty$, $\beta \ge 1 - 1/q$, and $N \in \mathbb{N}$; $N > 2(2s+3+\alpha)$. Then $f \in H\dot{K}^{\beta,p,N}_{\alpha,q}(\mathbb{R})$ if and only if there exist, for all $j \in \mathbb{N}\setminus\{0\}$, an $(\beta;q)$-atom a_j and $\lambda_j \in \mathbb{C}$, such that $\sum_{j=1}^\infty |\lambda_j|^p < \infty$ and $f = \sum_{j=1}^\infty \lambda_j a_j$. Moreover,

$$\|f\|_{H\dot{K}^{\beta,p,N}_{\alpha,q}(\mathbb{R})} = \inf\left(\sum_{j=1}^\infty |\lambda_j|^p\right)^{1/p}$$

where the infinimum is taken over all atomic decompositions of f . The main result of this subsection is the following theorem.

Theorem 3 Let $\alpha \ge \frac{-1}{2}$, $0 < p \leqslant 1 < q < \infty$, $\beta \ge 1 - 1/q$, and $N \in \mathbb{N}$; $N > 2(2s+3+\alpha)$.

(i) For $0 < p < 1$, let

$$C_{p,\sigma} = \int_{\mathbb{R}} t^{2(\alpha+1)(\beta-1+1/q)}\varphi(t)(1 + |\log_2|t||)^\sigma dt.$$

If for some $\sigma > \frac{1-p}{p}$, $C_p := C_{p,\sigma} < \infty$, then

$$\|\mathcal{H}_{\alpha,\varphi}(f)\|_{H\dot{K}^{\beta,p,N}_{\alpha,q}} \lesssim \|f\|_{H\dot{K}^{\beta,p,N}_{\alpha,q}}.$$

(ii) For $p = 1$, let

$$C_1 = \int_{\mathbb{R}} t^{2(\alpha+1)(\beta-1+1/q)}\varphi(t)dt.$$

If $C_1 < \infty$, then

$$\|\mathcal{H}_{\alpha,\varphi}(f)\|_{H\dot{K}^{\beta,1,N}_{\alpha,q}} \lesssim \|f\|_{H\dot{K}^{\beta,1,N}_{\alpha,q}}.$$

We now return to the example of the Dunkl-Cesàro operator $\mathcal{C}_{\alpha,\varphi}$.

Corollary 3 Let $\alpha \ge \frac{-1}{2}$, $0 < p \leqslant 1 < q < \infty$, $\beta \ge 1 - 1/q$, and $N \in \mathbb{N}$; $N > 2(2s+3+\alpha)$.

(i) For $0 < p < 1$, let

$$C_{p,\sigma} = \int_0^1 t^{2(\alpha+1)(\beta-1+1/q)}\varphi(t)(1 + |\log_2|t||)^\sigma dt.$$

If for some $\sigma > \frac{1-p}{p}$, $C_p := C_{p,\sigma} < \infty$, then

$$\|\mathcal{C}_{\alpha,\varphi}(f)\|_{H\dot{K}^{\beta,p,N}_{\alpha,q}} \lesssim \|f\|_{H\dot{K}^{\beta,p,N}_{\alpha,q}}.$$

(ii) For $p = 1$, let

$$C_1 = \int_0^1 t^{2(\alpha+1)(\beta-1+1/q)}\varphi(t)dt.$$

If $C_1 < \infty$, then

$$\|\mathcal{C}_{\alpha,\varphi}(f)\|_{H\dot{K}^{\beta,1,N}_{\alpha,q}} \lesssim \|f\|_{H\dot{K}^{\beta,1,N}_{\alpha,q}}.$$

Conclusion

We had introduced and studied recently in [4, 6, 7, 8] the Dunkl–Hausdorff operators on the weighted Lebesgue spaces $L^1_\alpha(\mathbb{R})$, Dunkl-type Hardy spaces $H^1_\alpha(\mathbb{R})$, Dunkl-type spaces of functions of bounded mean oscillation $BMO_\alpha(\mathbb{R})$, and Dunkl-type Sobolev spaces $W^{p,r}_\alpha(\mathbb{R})$. Gasmi and his collaborators in [10] introduced a new weighted Herz spaces associated with the Dunkl operator on $\mathbb{R}$. Also they characterize by atomic decompositions the corresponding Herz-type Hardy spaces. Motivated by this results, this paper aims to investigate the Dunkl–Hausdorff operators on these spaces in the spirit of those in [5]. As applications we provide necessary and sufficient conditions for Dunkl-Cesàro operator and sufficient conditions for Dunkl-Hardy operator to be bounded on the homogeneous weighted Herz space $\dot{K}^{\beta,p}_{\alpha,q}$.

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