On the non-commutative Kirillov formula

Tony Dooley
School of Mathematical and Physical Sciences
University of Technology Sydney

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Abstract

The Kirillov orbit method associates to a coadjoint orbit of a Lie group G an irreducible representation of the group. One can calculate the character of this representation from the Fourier transform of that orbit. However, if one starts not from the character, but from a general matrix entry, the Fourier transform is an interesting distribution on the Lie algebra. We can say a lot about the distributions which arise, their support, singular support and their convolutions. This work is joint with Raed Raffoul.

1 The Kirillov Character formula

The formula reads:

$$j(X)\xi_\lambda(\exp X) = \int_{\mathcal{O}_{\lambda+\delta}} e^{i\beta(X)} d\mu_{\lambda+\delta}(\beta) \ \ \forall X \in \mathfrak{g}.$$

where $\chi_\lambda = \mathrm{Tr}\pi_\lambda$.

This formula holds, in the sense of distributions, for a wide variety of Lie groups. For compact Lie groups, the formula is exact, as all irreducible representations are finite dimensional, all orbits are compact, and there is a canonical analytic real-valued square root j of the Jacobian of the exponential map.

For $\mathcal{SU}(2)$, it reads $\frac{sinx}{x} \times \frac{sin(nx)}{sinx} = \frac{sin(nx)}{x}$.

D. and Wildberger: the formula follows from a convolution formula for Ad-invariant distributions on the Lie algebra $\mathfrak{g}$. Specifically, one defines the **wrap** $\Phi\mu$ of such a distribution μ on $\mathfrak{g}$ by $< \Phi\mu, f > = < \mu, jf \circ \exp >$. Then one has

$$\Phi\mu *_G \Phi\nu = \Phi(\mu *_{\mathfrak{g}} \nu).$$

This is known as the **wrapping formula**.

We are interpreting the Fourier transform of a function or tempered distribution on $\mathfrak{g}$ as a function or tempered distribution on $\mathfrak{g}^*$.The Kirillov character formula says that the **Liouville measure** $\mu_{\lambda+\delta}$ on $\mathfrak{g}^*$ has a Fourier transform which is the C^∞ function $\chi_\lambda \circ \exp$ on $\mathfrak{g}$. Taking the inverse Fourier transform of each side, we may re-interpret this formula to say that the Fourier transform of the $C^\infty(\mathfrak{g})$ function $j\chi_\lambda \circ \exp$ equals the Liouville measure on $\mathcal{O}_{\lambda+\delta}$, that is

$$\int_{\mathfrak{g}} j\chi_\lambda(\exp X)e^{i\beta(X)}dX = \mu_{\lambda+\delta}(\beta). \tag{1}$$

2 The non-commutative Kirillov formula

We now wish to replace the character χ_λ in formula (1) with an arbitrary matrix coefficient of G. More specifically, let $\pi = \pi_\lambda$ be the irreducible representation of G corresponding to λ, acting on the finite dimensional Hilbert space $\mathcal{H}_\lambda$. Choose $\xi, \zeta \in \mathcal{H}_\lambda$, and consider the matrix coefficient

$$t^{(\lambda)}_{\xi,\zeta}(x) = < \xi, \pi(x)\zeta >, \quad x \in G. \tag{2}$$

Notice that if $\{\xi_i : i = 1, \dots, d_\pi\}$ is an orthonormal basis for $\mathcal{H}_\lambda$, we write $t^\lambda_{i,j} = t^{(\lambda)}_{\xi_i,\xi_j}$ and notice that $\chi_\lambda = \sum_{i=1}^{d_\pi} t^\lambda_{i,i}$.

Question What can be said about the distributional Fourier transforms of the functions $jt^\lambda_{i,j} \circ \exp$? That is, what is $\int_{\mathfrak{g}} j(X)t^\lambda_{i,j}(\exp X)e^{i\beta(X)}dX$ as a distribution on $\mathfrak{g}^*$?

As the Fourier transform of a C^∞ function, it is a distribution of compact support. In particular, we might ask to identify the support of this distribution. We can also ask what is the singular support of the distribution.

Franco Cazzaniga answered these questions for $SU(2)$ in 1992. Raed Raffoul's thesis (2007) generalised his results to arbitrary compact Lie groups. I suppose that some kind of generalisation holds for more general groups.

3 The Weyl calculus

Our results are based on Edward Nelson's formula for the Weyl calculus.

If $\mathcal{R}$ and $\mathcal{D}$ are Banach algebras with units $\mathbf{1}_{\mathcal{R}}, \mathbf{1}_{\mathcal{D}}$ respectively, and if E is a finite-dimensional subspace of $\mathcal{R}$ spanned by self-adjoint elements $\mathbf{1}_{\mathcal{R}}, x_1, \ldots, x_d$ and $\phi : E \to \mathcal{D}$ with $\|\phi\| = 1$ and if $\phi(\mathbf{1}_{\mathcal{R}}) = \mathbf{1}_{\mathcal{D}}$ is a linear map, then for all $x \in E$, $\phi(x)$ is an hermitian element of $\mathcal{D}$.

It follows that for any $f : \mathbb{R}^d \to \mathbb{C}$ with integrable Fourier transform, the following Bochner integral converges:

$$f_{\mathcal{R}}^{(\mathcal{D},\phi)}(x) := \int_{\mathbb{R}^d} \hat{f}(\lambda) e^{i\lambda\cdot\phi(x)} d\lambda. \tag{3}$$

If f belongs to the Schwarz space $\mathcal{S}(\mathbb{R}^d)$, then $\langle W_{\mathcal{R}}^{(\mathcal{D},\phi)}(x), f\rangle = f_{\mathcal{R}}^{(\mathcal{D},\phi)}(x)$ is well-defined, and so defines a $\mathcal{D}$-valued tempered distribution $W_{\mathcal{R}}^{(\mathcal{D},\phi)}(x)$, known as the **Weyl calculus** of x with respect to $\mathcal{D}$ and ϕ.

For the special case where $\mathcal{D}$ is the set $\mathcal{L}(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$, E is a finite subalgebra of $\mathcal{L}(\mathcal{H})$ generated by a finite set of self adjoint operators and ϕ is the identity, Edward Nelson gave an explicit formula for the Weyl calculus [12]. (See next slide)

We may assume that $\mathcal{H}$ is an d-dimensional Hilbert space, and $A = (A_1, \ldots, A_n)$ an n-tuple of self-adjoint operators on $\mathcal{H}$. Let Σ_d be the unit ball in $\mathcal{H}$, and ν_d the unitary invariant probability measure on Σ_d.

Define the **joint numerical range** of A by:

$$W_A(u) = (\langle A_1 u, u\rangle, \ldots, \langle A_n u, u\rangle), \;\; u \in \Sigma_d \tag{4}$$

and define $\mu_A = \nu \circ W_A^{-1}$. This is a measure on $\mathbb{R}^n$ supported on the inverse image under W_A of the unit sphere Σ_d. Given a basis of $\mathcal{H}$,we may represent the A_i as $d \times d$ matrices. We let $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ be the operators of partial differentiation on $\mathbb{R}^n$, set $A \cdot \frac{\partial}{\partial x} = A_1 \frac{\partial}{\partial x_1} + \ldots A_n \frac{\partial}{\partial x_n}$.

For a matrix differential operator M on $\mathbb{R}^d$, let $\phi_j(M)$ denote the sum of the principal minors of M of order j.

Theorem 1 *Let $A = (A_1, \ldots, A_n)$ be as above. Then the Weyl calculus for A is given by:*

$$W_{\mathcal{L}(\mathcal{H})}(A) = \sum_{k=0}^{n-1} \sum_{j=0}^{n-k-1} \sum_{m=0}^{j} \binom{j}{m} \frac{(n-1)!}{(n-j+m-1)!} (A\cdot\frac{\partial}{\partial x})^k \phi_{n-j-k-1}(A\cdot\frac{\partial}{\partial x})(\frac{\partial}{\partial x}\cdot x)^m \mu_A.$$

We apply Theorem 1 to the case where $\pi = \pi_\lambda$ is an irreducible representation of G acting in $\mathcal{H}_\lambda$, $X = (X_1, \ldots, X_n)$ is a basis for $\mathfrak{g}$ and we set $A_i = \frac{1}{i} d\pi_\lambda(X_i)$. In this case, we simplify our notation by writing W_λ instead of $W_{\mathcal{L}(\mathcal{H}_{\pi_\lambda})}(X)$.

The joint numerical range defined in (4) may be thought of as a mapping from the unit ball Σ of $\mathcal{H}_\lambda$ to $\mathfrak{g}^*$, if we define $W_X(u) = \sum_{i=1}^n \alpha_i \langle A_i u, u\rangle = \langle \frac{1}{i} d_{\pi_\lambda}(X) X u, u\rangle$, when $X = \sum \alpha_i X_i \in \mathfrak{g}$. The map $u \mapsto W_{\cdot}(u)$ is then nothing but the moment map of π(see [15]).

An easy calculation then shows that $W_X(\pi(g)u) = W_{\mathrm{Ad}(g)X}(u)$, i.e. $W(\pi(g)u) = W(u) \circ \mathrm{Ad}^*(g)$. Now for $H \in \mathfrak{t}$, we have $W_H(u) = \langle \pi(H)u, u\rangle$, and so if u_λ belongs to the space V_λ in $\mathcal{H}$, we have $W_H(u_\lambda) = \lambda(H)$. Similarly for the weights ν of π, the we have $W_H(u_\nu) = \nu(H)$. It follows that convex hull of the image of W_λ is exactly the orbit $\mathcal{O}_\lambda$.

Re-examining the definition of the Weyl calculus, (3), as above, we identify $\mathbb{R}^n$ with $\mathfrak{g}^*$, so that we can write, for $\phi \in \mathcal{S}(\mathfrak{g}^*)$

$$\langle W_\lambda, \phi\rangle = \int_{\mathfrak{g}} \hat{\phi}(X) e^{d\pi(X)} dX \tag{5}$$

$$= \int_{\mathfrak{g}} \hat{\phi}(X) \pi_\lambda(\exp X) dX \tag{6}$$

In these coordinates, W_λ is an $\mathcal{L}(\mathcal{H}_\lambda)$-valued distribution on $\mathfrak{g}^*$. As above, it is easy to see that for $g \in G$,

$$W_\lambda \circ \mathrm{Ad}^*(g) = \pi_\lambda(g^{-1}) W_\lambda \pi_\lambda(g). \tag{7}$$

It follows that W_λ is determined by its restriction to the Cartan subalgebra, $\mathfrak{t}$.

Of course, the elements of $W_{\mathcal{L}(\mathcal{H}_\lambda)}$ are nothing but $d_\pi \times d_\pi$ matrices. We see by equation (5) that its entries are exactly the Fourier transforms of the matrix coefficients $t^\lambda_{\xi,\zeta}(\exp X)$ considered as distributions on $\mathfrak{g}$.

By Theorem 1, we may re-express these distributions as certain matrix-valued derivatives of the Liouville measure μ_λ.

4 Supports and singular supports

Since the support of μ_λ is the orbit $\mathcal{O}_\lambda$, the last observation of the previous section gives immediately

Theorem 2 *(i) The support of W_λ is the interior of $\mathcal{O}_\lambda$*

(ii) The singular support of W_λ is $\mathcal{O}_\lambda$.

The above formulae shows how to compute the Fourier transform of the matrix entries composed with exp, $t^\lambda_{\xi,\zeta} \circ \exp$. Both the original Kirillov formula and the non-commutative Kirillov formula proved in [3] and extended in [13], however, concern the Fourier transforms of the functions $j(X)t^\lambda_{\xi,\zeta} \circ \exp(X)$.

Now j has distributional Fourier transform μ_δ, the Liouville measure on the orbit through $\delta = \frac{1}{2}\sum_{\alpha\succ 0}\alpha$.

It follows that the Fourier transform of $j(X)t^\lambda_{\xi,\zeta} \circ \exp(X)$ is a distribution on $\mathfrak{g}^*$ which is given by $\mu_\delta * \left(W_\lambda\right)_{\xi,\zeta}$. As convolutions and differentiation commute, we get

Theorem 3

$$j.\widehat{t^\lambda_{\xi,\zeta} \circ \exp} = \sum_{k=0}^{n-1}\sum_{j=0}^{n-k-1}\sum_{m=0}^{j}\begin{pmatrix} j \\ m \end{pmatrix}\frac{(n-1)!}{(n-j+m-1)!}(A\cdot\frac{\partial}{\partial x})^k\phi_{n-j-k-1}(A\cdot\frac{\partial}{\partial x})(\frac{\partial}{\partial x}\cdot x)^m\mu_\delta * \mu_\lambda\mid_{\xi,\zeta}.$$

Repka Wildberger and I gave explicit expressions for the convolutions of adjoint orbits. From these, it can be seen that the support of $\mu_\delta * \mu_\lambda$ intersects $\mathfrak{t}^*$ in the convex hull of the points $\{\lambda + w\delta : w \in W\}$. Further, the convolution is a piecewise polynomial expression with discontinuities only on the orbits of the $|W|$ points $\{\lambda + w\delta : w \in W\}$. Thus we have:

Corollary 4 *[13] The support of the distribution $j.\widehat{t^\lambda_{\xi,\zeta} \circ \exp}$ is an Ad^*-invariant set whose intersection with $\mathfrak{t}^{*+}$ is the convex hull of $\{\lambda + w\delta : w \in W\}$. Further, its singular support is the union of the coadjoint orbits $\bigcup_{w\in W}\mathcal{O}_{\lambda+w\delta}$.*

This theorem generalizes Cazzaniga's Theorem 1.

5 A formula for transforms of matrix coefficients

Theorem 3 can be slightly reformulated. Let us equip $\mathfrak{g}$ with the norm $|\cdot| = \sqrt{-\kappa(\cdot,\cdot)}$, and choose a basis $\{X_1, \ldots, X_n\}$. Now equip $\mathfrak{g}^*$ with the dual norm, choose the dual basis $\{x_i : i = 1, \ldots, n\}$ for $\mathfrak{g}^*$, and set $r = (|x_1|^2 + \cdots + |x_n|^2)^{\frac{1}{2}}$. Then we have:

$$\frac{\partial}{\partial x} \cdot x = r\frac{\partial}{\partial r} + nI \tag{8}$$

This observation allows one to see that the matrix entries of the distribution $\hat{j} * W_\lambda(X)$ are polynomials of degree at most $(d_\lambda)^{n-1}$ acting on $\mu_\delta * \mu_\lambda$.

Consider $G = \mathcal{SU}(2)$, and choose the standard basis of Pauli spin matrices $A_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ $A_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Then one calculates easily that $A \cdot \frac{\partial}{\partial x} = \begin{pmatrix} i\frac{\partial}{\partial x_1} & -\frac{\partial}{\partial x_2} + i\frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_2} + i\frac{\partial}{\partial x_3} & -i\frac{\partial}{\partial x_1} \end{pmatrix}$ and that $A \cdot \frac{\partial^2}{\partial x^2} = \begin{pmatrix} -\triangle & 0 \\ 0 & -\triangle \end{pmatrix}$ where

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

is the Laplacian on $\mathfrak{g}^*$.

Notice that $A \cdot \frac{\partial}{\partial x} = \begin{pmatrix} H^* & L_\alpha^* \\ L_{-\alpha}^* & -H^* \end{pmatrix}$ where H spans $\mathfrak{t}$, L_α and $L_{-\alpha}$ are the positive and negative root directions and the stars denote the dual basis elements.

It follows that the Weyl calculus is given by

$$W_L(A) = \big((1 - \triangle)I + \begin{pmatrix} H^* & L_\alpha^* \\ L_{-\alpha}^* & -H^* \end{pmatrix} (3I + r\frac{\partial}{\partial r})\big)\mu_A$$

and that the transforms of the matrix entries are given by:

$$j(X)\widehat{t_{ij}^{(\lambda)}(\exp X)} = \big((1 - \triangle)I + \begin{pmatrix} H^* & L_\alpha^* \\ L_{-\alpha}^* & -H^* \end{pmatrix} (3I + r\frac{\partial}{\partial r})\big)\mu_\delta * \mu_\lambda|_{ij}. \tag{9}$$

For $\mathcal{SU}(2)$, $\mu_\delta * \mu_\lambda$ is a radial function whose restriction to $\mathfrak{t}^+ = \mathbb{R}^+$ is exactly the characteristic function of the interval $[\lambda - 1, \lambda + 1]$. Thus we see that $\frac{\partial}{\partial r}\mu_\delta * \mu_\lambda$ is the difference of two delta functions on the orbits: $\mu_{\lambda+1} - \mu_{\lambda-1}$. *Similar formulae hold for general compact Lie groups.*

6 Convolution laws

Does the wrapping formula generalise to the non-commutative setting?

Following Anderson's treatment of the nilpotent case, Dominique Manchon defines, for a solvable group, and for $p, q \in C^\infty(\mathfrak{g}^*)$,

$$W_\pi(p\sharp q) = W_\pi(p) \circ W_\pi(q).$$

Then we define, for p as above,

$$\tilde{p}(x) = j^{-2}(\log x)\hat{p} \circ \log(x)$$

and notice that

$$p\sharp q = \tilde{p} * \tilde{q},$$

where $*$ denotes convolution in G. This equation presumes that the supports of $\hat{p}$ and $\hat{q}$ are such that all the formulae are defined.

The first of these equations is really nothing but the fact that the group Fourier transform maps group convolution to composition of operators.

Now we re-write this formula somewhat, starting from a tempered distribution μ on $\mathfrak{g}$, we note that $\hat{\mu}$ is a distribution on $\mathfrak{g}^*$ and the convolution $p = \mu_\delta * \hat{\mu}$ exists as a tempered distribution on $\mathfrak{g}^*$, and its $\mathfrak{g}^*$ Fourier transform is therefore a tempered distribution on $\mathfrak{g}$. If μ is locally in L^1 then this Fourier transform is also locally L^1 and has functional form $\hat{p}(X) = j(X)\mu(X)$.

If the support of μ satisfies Manchon's conditions, then one has $\tilde{p}(x) = \frac{\mu(\log(x))}{j(\log(x)}$ for $x \in G$. We claim that

$$\tilde{p} = \Phi(\mu)$$

, where Φ is the wrapping map.

For $f \in C^\infty(G)$, we have

$$\begin{aligned} < \Phi\mu, f > &= < \mu, jf \circ \exp > \\ &= \int_{\mathfrak{g}} \mu(X) j(X) f(\exp X) dX \quad (10) \\ &= \int_{\mathfrak{g}} \frac{\mu(X)}{j(X)} f(\exp X) j(X)^2 dX \quad (11) \\ &= \int_G \tilde{p}(x) f(x) dx. \quad (12) \end{aligned}$$

It follows from Manchon's formula (6) that

$$\Phi\mu *_G \Phi\nu = (\mu_\delta * \hat{\mu})\sharp(\mu_\delta * \hat{\nu}) \tag{13}$$

for distributions $\mu, \nu \in \mathcal{S}(\mathfrak{g})$.

Actually, it can be shown that this formula continues to hold without Manchon's conditions on the supports of μ and ν. **Note:** These functions are not necessarily Ad^*-invariant!

It is thus of interest to calculate the right hand side of (13), as we can then calculate G-convolutions in terms of $\mathfrak{g}^*$.

Now we notice that the right hand side of (3) has the form

$$\frac{1}{i} d\pi_\lambda p(X_1, \ldots, X_n)\mu_\delta * \mu_\lambda \tag{14}$$

where p is the $\mathfrak{g}$-valued differential operator on $\mathfrak{g}^*$ given by

$$p(X_1, \ldots, X_n) = \sum_{k=0}^{n-1} \sum_{j=0}^{n-k-1} \sum_{m=0}^{j} \begin{pmatrix} j \\ m \end{pmatrix} \frac{(n-1)!}{(n-j+m-1)!} (X \cdot \frac{\partial}{\partial x})^k \phi_{n-j-k-1}(X \cdot \frac{\partial}{\partial x})(\frac{\partial}{\partial x} \cdot x)^m.$$

If we test the distribution W_λ against a test function $\varphi \in \mathcal{S}(\mathfrak{g}^*)$, we get

$$< W_\lambda, \varphi >= \frac{1}{i} d\pi_\lambda \int_{\mathfrak{g}^*} p(X_1, \ldots, X_n)\varphi d\mu_\lambda$$

.

It follows that for μ, $\nu \in \mathcal{S}(\mathfrak{g}^*)$,

$$\Phi\mu *_G \Phi\nu = \widehat{p(.)\widehat{j\mu}} *_{\mathfrak{g}} j\nu.$$

In the case where μ and ν are both Ad-invariant, one sees easily that this formula reduces to the wrapping theorem.

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