New Function Spaces Associated to Representations of Nilpotent Lie Groups and Generalized Time-Frequency Analysis

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Motivation and Goal

- Explicit examples of new function spaces associated to low-dimensional *nilpotent* Lie groups (coorbit spaces).

- Why are coorbit spaces for different groups *different*?

- "Precisely how can we prove the distinctness" [of different families of function spaces]? [Fischer,Rottensteiner, Ruzhansky]. "...invest effort ... in concreteness" and "to compare modulation spaces with other function spaces" [Mantoiu]

Coorbit Spaces: Set-up

- G locally compact group with Haar measure dx
- $\pi : G \to \mathcal{U}(\mathcal{H})$ irreducible, unitary, *square-integrable*

representation on Hilbert space $\mathcal{H}$

$$\int_G \langle f_1, \pi(x) g_1 \rangle \overline{\langle f_2, \pi(x) g_2 \rangle} \, dx = d_\pi^{-1} \langle f_1, f_2 \rangle \overline{\langle g_1, g_2 \rangle}$$

for all $f_1, f_2, g_1, g_2 \in \mathcal{H}$.
d_π formal dimension

- Fix a non-zero $g \in \mathcal{H}$:

$$V_g^\pi f(x) = \langle f, \pi(x) g \rangle \qquad x \in G$$

maps elements $f \in \mathcal{H}$ to functions on G.
Coherent state transform, generalized wavelet transform, or short-time Fourier transform.

Coorbit spaces: definition

Define norm by pullback from G to $\mathcal{H}$

$$\|f\|_{Co_\pi L^p} = \left(\int_G |\langle f, \pi(x)g \rangle|^p \, dx \right)^{1/p} = \|V_g^\pi f\|_{L^p(G)} \, .$$

$Co_\pi L^p(G)$ is completion of test functions, e.g., of C^∞-vectors $\mathcal{H}_\pi^\infty$, with respect to $\| \cdot \|_{Co_\pi L^p}$ for $p < \infty$. For $p = \infty$ some weak completion.

Coorbit spaces: definition

Define norm by pullback from G to $\mathcal{H}$, $m \geq 0$ suitable weight

$$\|f\|_{Co_\pi L^p_m} = \Big(\int_G |\langle f, \pi(x)g\rangle|^p \, m(x)^p \, dx \Big)^{1/p} = \| V_g^\pi f \|_{L^p_m(G/Z)} \, .$$

$Co_\pi L^p_m(G)$ is completion of test functions, e.g., of C^∞-vectors $\mathcal{H}_\pi^\infty$, with respect to $\| \cdot \|_{Co_\pi L^p}$ for $p < \infty$. For $p = \infty$ some weak completion.

Many generalizations:

- mixed-norm L^p-spaces, $p < 1$, solid function spaces on G
- reducible representations
- various spaces of test functions and distributions

Feichtinger, KG 1987-91

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Modification for nilpotent groups

From now on G is simply connected *nilpotent* Lie group with center Z.

Assumption: $(\pi, \mathcal{H})$ is square-integrable modulo the center
$\Rightarrow \quad \pi|_Z = \chi(z)\mathrm{I}_{\mathcal{H}}$ and $x \in G \mapsto |\langle f, \pi(zx)g\rangle|$ defined on G/Z. [1]

Norm on coorbit space $Co_\pi L^p_m(G/Z)$

$$\|f\|_{Co_\pi L^p_m} = \left(\int_{G/Z} |\langle f, \pi(\dot{x})g\rangle|^p \, m(\dot{x})^p \, d\dot{x}\right)^{1/p} = \|V_g^\pi f\|_{L^p_m(G/Z)} \, .$$

[1] Dilemma: either start with representations and work on G/Z or use projective representations

Coorbit spaces: general properties

Proposition

1. $Co_\pi L^p_m(G/Z)$, *is a Banach space for* $1 \leq p \leq \infty$.
2. *Invariance:* $Co_\pi L^p_m(G/Z)$ *is invariant with respect to the representation* π*, in particular,* $\|\pi(y)f\|_{Co_\pi L^p} = \|f\|_{Co_\pi L^p}$.
3. *If* $h \in Co_\pi L^1(G/Z)$*, then* $\|V_h^\pi f\|_{L^p}$ *is an equivalent norm on* $Co_\pi L^p(G/Z)$.
4. *Duality:* $(Co_\pi L^p(G/Z))^* = Co_\pi L^{p'}(G/Z)$ *for* $1 \leq p < \infty, p' = \frac{p}{p-1}$ *with duality* $\langle f, h\rangle = \int_{G/Z} V_g^\pi f(\dot{x}) \overline{V_g^\pi h(\dot{x})}\, d\dot{x}$.
5. $Co_\pi L^p_m(G/Z)$ *is isomorphic to* ℓ^p.
6. *For* $m \geq 1$ *and* $1 \leq p \leq 2$ $Co_\pi L^p_m(G/Z) \subseteq Co_\pi L^2(G/Z) = \mathcal{H}$.

Main result: atomic decompositions, frames

Examples

- Coorbit spaces of $ax + b$-group lead to Besov-Triebel-Lizorkin spaces
- Coorbit spaces of semisimple Lie groups $\mathrm{SU}(n, 1)$ lead to Bergman spaces.
- Other extensions of $\mathbb{R}^d$, $G = \mathbb{R}^d \ltimes H$, lead to shearlet spaces etc.
- Heisenberg group leads to modulation spaces

Modulation Spaces

Translation $T_x f(t) = f(t-x)$
Modulation (frequency shift) $M_\xi f(t) = e^{2\pi i \xi \cdot t} f(t)$
Short-time Fourier transform $S_g f(x,\xi) = \langle f, M_\xi T_x f\rangle$ for fixed window g in $L^2(\mathbb{R}^d)$ or $\mathcal{S}(\mathbb{R}^d)$.

Heisenberg group $\mathbb{H}_d \cong \mathbb{R}^{2d} \times \mathbb{R}$ with
$(x,y,z)\cdot(u,v,w) = (x+u, y+v, z+w+x\cdot v)$
with Schrödinger representation on $f \in L^2(\mathbb{R}^d)$.

$$\pi_\lambda(x,y,z)g(t) = e^{2\pi i\lambda z} e^{-2\pi i \lambda y\cdot t} f(t-x) = e^{2\pi i \lambda z} M_{-\lambda y} T_x g(t)$$

leads to *modulation spaces*

$$\|f\|^p_{Co_\pi L^p_m} = \int_{\mathbb{R}^{2d}} |V_g^{\pi_\lambda} f(x,y)|^p m(x,y)^p \, dxdy$$
$$= \int_{\mathbb{R}^{2d}} |\langle f, M_{-\lambda y} T_x g\rangle|^p m(x,y)^p \, dxdy := \|f\|^p_{M^p_{m_\lambda}}$$

$G_{6,16} \cong \mathbb{R}^6$

$$(x_1, \dots, x_6) \cdot (y_1, \dots, y_6) =$$
$$(x_1 + y_1 + x_5 y_3 + x_6 y_4, x_2 + y_2 + x_6 y_5, x_3 + y_3, x_4 + y_4, x_5 + y_5, x_6 + y_6)$$

Irreducible square-integrable representation on $L^2(\mathbb{R}^2)$
$(\lambda \neq 0, \mu \in \mathbb{R})$

$$\pi_{\lambda,\mu}(x_1, \dots, x_6)g(s,t) =$$
$$\exp 2\pi i\Big(\lambda(x_1 - x_3 s - x_4 t) + \mu(x_2 - x_5 x_6 + x_6 s)\Big) g(s - x_5, t - x_6)\,.$$

$$\pi_{\lambda,\mu}(\dot{x}) = M_{(-\lambda x_3 + \mu x_6, -\lambda x_4)} T_{(x_5, x_6)}$$
$$V_g^{\pi_{\lambda,\mu}} f(\dot{x}) = \langle f, M_{(-\lambda x_3 + \mu x_6, -\lambda x_4)} T_{(x_5, x_6)} g \rangle$$
$$\|f\|_{Co_\pi L^p(G_{6,16}/Z)} = \|f\|_{M^p(\mathbb{R}^2)}$$

Nothing new: $G_{6,16}/Z \cong \mathbb{R}^4 \cong \mathbb{H}_2/Z$.

$G_{5,3} \cong \mathbb{R}^5$

$$[X_5, X_4] = X_2, \quad [X_5, X_2] = X_1, \quad [X_4, X_3] = X_1 \,.$$

with multiplication

$$(x_1, \dots, x_5) \cdot (y_1, \dots, y_5) =$$
$$(x_1 + y_1 + x_4 y_3 + x_5 y_2 + \tfrac{1}{2} x_5^2 y_4, x_2 + y_2 + x_5 y_4, x_3 + y_3, x_4 + y_4, x_5 + y_5) \,.$$

and irreducible square-integrable representation

$$\pi_\lambda(x) g(s,t) = \exp\Big(2\pi i \lambda (x_1 - x_3 x_4 + x_4 s - x_2 t + \tfrac{1}{2} x_4 t^2)\Big)\, g(s - x_3, t - x_5) \,.$$

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and irreducible square-integrable representation

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Translation $T_{(x_3, x_5)}$

$G_{5,3} \cong \mathbb{R}^5$

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and irreducible square-integrable representation

$$\pi_\lambda(x) g(s,t) = \exp\Big(2\pi i \lambda (x_1 - x_3 x_4 + x_4 s - x_2 t + \tfrac{1}{2} x_4 t^2)\Big)\, g(s - x_3, t - x_5) \,.$$

Modulations $M_{(\lambda x_4, -\lambda x_2)}$

$G_{5,3} \cong \mathbb{R}^5$

$$[X_5, X_4] = X_2, \quad [X_5, X_2] = X_1, \quad [X_4, X_3] = X_1 \, .$$

with multiplication

$$(x_1, \dots, x_5) \cdot (y_1, \dots, y_5) =$$

$$(x_1 + y_1 + x_4 y_3 + x_5 y_2 + \tfrac{1}{2} x_5^2 y_4, x_2 + y_2 + x_5 y_4, x_3 + y_3, x_4 + y_4, x_5 + y_5) \, .$$

and irreducible square-integrable representation

$$\pi_\lambda(x) g(s,t) = \exp\Big(2\pi i \lambda (x_1 - x_3 x_4 + x_4 s - x_2 t + \tfrac{1}{2} x_4 t^2)\Big) \, g(s - x_3, t - x_5) \, .$$

"Chirp", quadratic character $\boxed{\mathcal{N}_u f(s,t) = e^{\pi i u t^2} f(s,t)}$

Coorbit spaces $Co_\pi L^p(G_{5,3}/Z)$?

Are they "new"? How to distinguish from modulation spaces?

Coorbit spaces with respect to different groups have different invariance properties and therefore should be "different".

For function spaces on $\mathbb{R}^d$: $B_1(\mathbb{R}^d) \neq B_2(\mathbb{R}^d)$.
Need to construct element in $B_1 \setminus B_2$ or $B_2 \setminus B_1$

Chirps on modulation spaces

Let $C = C^T$ be a real-valued symmetric $d \times d$-matrix and

$$\mathcal{N}_C f(t) = e^{-i\pi Ct\cdot t} f(t)$$

Let $D = (4\mathrm{I} + C^2)^{-1}$ $\Delta = \begin{pmatrix} 2D & -DC \\ -DC & \mathrm{I}-2D \end{pmatrix}$ acting on $z \in \mathbb{R}^{2d}$.

Proposition

Let $\phi(t) = e^{-\pi t\cdot t}$ (Gaussian on $\mathbb{R}^d$) and $1 \le p \le 2$. Then

$$|\langle \mathcal{N}_C\phi, M_\xi T_x\phi\rangle| = \det(4\mathrm{I} + C^2)^{-1/4} e^{-\pi\Delta(\xi,x)^T\cdot(\xi,x)^T} ,$$

and

$$\|\mathcal{N}_C\phi\|_{M^p(\mathbb{R}^d)} = \det(4\mathrm{I} + C^2)^{\frac{1}{2p}-\frac{1}{4}} . \qquad (1)$$

Proof via Gaussian integrals (Folland, Cordero-Nicola).
Insight: modulation spaces are invariant under chirps, but $\mathcal{N}_C$ is no isometry.

Main observation

Proposition

Let $p, q \in [1, \infty], p \neq 2$. Then

$$Co_\pi L^p(G_{5,3}/Z) \neq M^q(\mathbb{R}^2)\,.$$

Proof sketch: Recall for $\lambda = 1$ and $\pi_1 = \pi$ and $C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$\pi(0,0,x_4,0)g(s,t) = e^{2\pi i x_4 s + \pi i x_4 t^2} g(s,t) = M_{(x_4,0)} \mathcal{N}_{x_4 C} g(s,t)$$

By Proposition 1

$$\|\pi(0,0,x_4,0)f\|_{Co_\pi L^p} = \|f\|_{Co_\pi L^p}\,.$$

By Proposition 2 for Gaussian $\phi(s,t) = e^{-\pi(s^2+t^2)}$ and $\det(4\mathrm{I} + x_4^2 C^2) = 4 + x_4^2$

$$\begin{aligned} \|\pi(0,0,x_4,0)\phi\|_{M^p(\mathbb{R}^2)} &= \|M_{(x_4,0)}\mathcal{N}_{x_4 C}\phi\|_{M^p(\mathbb{R}^2)} \\ &= c_p(4+x_4^2)^{\frac{1}{2p}-\frac{1}{4}} \\ &\asymp |x_4|^{\frac{1}{p}-\frac{1}{2}} . \end{aligned}$$

$\Rightarrow$ one-parameter group $\pi(0,0,0,x_4,0)$ is unbounded on $M^p(\mathbb{R}^2)$ for $1 \le p < 2$.

Conclusion: $Co_\pi L^p(G_{5,3}/Z) \neq M^p(\mathbb{R}^2)$.

For $p > 2$ use duality. [2] ■

In fact: $\boxed{Co_\pi L^p(G_{5,3}/Z) \neq M^p_m(\mathbb{R}^2)}$ for arbitrary weights.

[2] $p = 2$: $Co_\pi L^2(G/Z) = \mathcal{H} = L^2$ independent of representation.

$$G_{6,19} \cong \mathbb{R}^6$$

$$(x_1, \ldots, x_6) \cdot (y_1, \ldots, y_6) = (x_1 + y_1 + x_6 y_3,$$
$$x_2 + y_2 + x_5 y_4 + x_5 x_6 y_5 + \tfrac{1}{2} x_6 y_5^2, x_3 + y_3, x_4 + y_4 + x_6 y_5, x_5 + y_5, x_6 + y_6)$$

with square-integrable representation modulo Z

$$\pi_{\lambda,\mu}(x_1, \ldots, x_6) g(s,t) =$$
$$e^{2\pi i\left(\lambda(x_1 - x_3 t) + \mu(x_2 - \frac{1}{2} x_5^2 x_6 - x_4 s + x_5 x_6 s - \frac{1}{2} x_6 s^2)\right)} g(s - x_5, t - x_6)\,.$$

Proposition

Let $\lambda\mu \neq 0$ and $p, q \in [1, \infty], p \neq 2$. Then
(i) $Co_{\pi_{\lambda,\mu}} L^p(G_{6,19}/Z) \neq M^q(\mathbb{R}^2)$, and
(ii) $Co_{\pi_{\lambda,\mu}} L^p(G_{6,19}/Z) \neq Co_\pi L^q(G_{5,3}/Z)$.

The Dynin-Folland group

$$\Big(\pi(z, y_1, y_2, y_3, x_1, x_2, x_3)f\Big)(t_3, t_2, t_1) =$$
$$e^{2\pi i(z+\sum_{k=1}^{3} t_j y_j - t_1 t_2 y_3/2)} f(t_3 + x_1 + t_1 x_2, t_2 + x_2, t_1 + x_3)\,.$$

Proposition (Fischer, Rottensteiner, Ruzhansky)

For $p \in [1, \infty], p \neq 2$ we have $Co_\pi L^p(G/Z) \neq M^p(\mathbb{R}^3)$.

Compare: Proof via Voigtländer's theory of decomposition spaces ($>$ 180 pages) or one page proof via chirps.

Why generalized time-frequency analysis?

Square-integrable representation of nilpotent Lie group can be written roughly as

$$g(t) \to \pi(x,\xi)g(t) = e^{2\pi i P(t,\xi)} g(tx)\,,$$

with $t \mapsto t\,x$ a group action (translation) and $e^{2\pi i P(t,\xi)}$ a generalized modulation (with P a polynomial) and x,ξ suitable partition of coordinates of G/Z in "time" and "frequency".

$$V_g^\pi(\dot{x}) = \langle f, \pi(\dot{x})g\rangle$$

is a generalized short-time Fourier transform on some phase space.

Background: quantization on flat orbits etc. [Beltita,Mantoiu]

Conclusion

- Study more explicit examples
- Applications: concrete operators via representations $\int_G f(x)\pi(x)dx$ (quantization)
- Guess: $G_1/\ker\pi_1 \cong G_2/\ker\pi_2$ implies $Co_{\pi_1}(G_1) \cong Co_{\pi_2}(G_2)$, examples suggest a converse.

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