Coarse geometric methods for generalized wavelet approximation theory

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Outline

1. Generalized wavelet systems
2. Approximation theory of generalized wavelet systems
3. A metric criterion for classification
4. Some applications

Overview

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2. Approximation theory of generalized wavelet systems
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Generalized Wavelet Systems: Definition

Definition

A GWS is a family $\Psi = (\psi_j)_{j \in J} \subset \mathrm{L}^2(\mathbb{R}^d)$. Ψ is called reproducing if it fulfills

$$\forall f \in \mathrm{L}^2(\mathbb{R}^d) : \|f\|_2^2 = \sum_{j \in J} \|f * \psi_j\|_2^2 \ .$$

Theorem (Calderón condition)

A system $\Psi = (\psi_j)_{j \in J}$ is reproducing iff

$$\sum_{j \in J} |\widehat{\psi}_j(\xi)|^2 = 1 \ \text{ a.e.}$$

holds. In this case,

$$f = \sum_{j \in J} f * \psi_j * \psi_j^* \ , \forall f \in \mathrm{L}^2(\mathbb{R}^d).$$

Examples of one-dimensional systems

One-dimensional wavelet systems

Fix a suitable $\psi \in \mathrm{L}^2(\mathbb{R})$, $J = \mathbb{Z}$, and define $\psi_j(x) = 2^{-j}\psi(2^{-j}x)$.

One-dimensional Gabor systems

Fix a suitable $\psi \in \mathrm{L}^2(\mathbb{R})$, $J = \mathbb{Z}$, and define $\psi_j(x) = \exp(2\pi ijx)\psi(x)$.

Other types of systems in dimension one

ERB-lets (acoustics), multi-window Gabor systems

Examples of two-dimensional systems

Oriented wavelet systems

Fix a suitable $\psi \in \mathrm{L}^2(\mathbb{R}^2)$, $J = \mathbb{Z} \times \{1, \ldots, K\}$, and define $\psi_{n,k}(x) = 2^{-n}\psi(2^{-n}R_k x)$, where

$$R_k = \begin{pmatrix} \cos(2\pi k/K) & \sin(2\pi k/K) \\ -\sin(2\pi k/K) & \cos(2\pi k/K) \end{pmatrix} .$$

The associated system is called an oriented wavelet system.

Gabor systems

Fix a suitable $\psi \in \mathrm{L}^2(\mathbb{R}^2)$, $J = \mathbb{Z}^2$, and define $\psi_j(x) = \exp(2\pi i \langle j, x\rangle)\psi(x)$.

Shearlets

Shearlets

Fix a suitable $\psi \in L^2(\mathbb{R}^2)$, let $J = \mathbb{Z}^2$, and define

$$\psi_{n,k}(x) = 2^{\frac{3}{2}n}\psi(D_{2^n}S_k x) \ ,$$

where

$$S_k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \ , \ D_a = \begin{pmatrix} a & 0 \\ 0 & a^{1/2} \end{pmatrix} \ .$$

The associated system is called a shearlet system.

Related constructions

Curvelets, cone-adapted shearlets, higher-dimensional shearlet systems.

Anisotropic wavelet systems

Definition

A matrix $A \in \mathrm{GL}(d, \mathbb{R})$ is called expanding if $|\lambda| > 1$ holds for all eigenvalues of λ of A.

Definition

Let A be an expanding matrix. An A-wavelet is a function $\psi \in \mathrm{L}^2(\mathbb{R}^d)$ such that

$$\psi_j(x) = |\det(A)|^{-j} \psi(A^{-j} x)$$

defines a reproducing system Ψ. This system Ψ is called homogeneous anisotropic wavelet system associated to A.
The inhomogeneous wavelet system is given by

$$\{\varphi\} \cup \{\psi_j\}_{j \in \mathbb{N}}$$

with suitably chosen φ.

Filterbank interpretation

GMS as filterbanks

- Let $\Psi = (\psi_j)_{j \in J}$ be reproducing.
- Convolution theorem: $(f * \psi_j)^\wedge = \widehat{f} \circ \widehat{\psi_j}$.
- Signal processing view: ψ_j corresponds to a channel in a filter bank.
- Typically: $\widehat{\psi_j}$ concentrated in "frequency band" $\Omega_j \subset \mathbb{R}^d$.
- The channel output $(f * \psi_j)^\wedge$ is concentrated in Ω_j , the contribution of frequencies in Ω_j to f in the sum.
- The reconstruction formula

$$f = \sum_{j \in \mathbb{J}} f * \psi_j * \psi_j^* \ .$$

means that f can be perfectly reconstructed from the channel contributions $f * \psi_j$.

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Besov-type decomposition spaces

Definition (Feichtinger/Gröbner; Nielsen; Voigtlaender)

Let $\Psi = (\psi_j)_{j \in J}$ denote a GWS, and $\mathcal{P} = (P_j)_{j \in J}$ a collection of open, relatively compact sets. We assume the GWS is subordinate to $\mathcal{P}$, i.e., $\mathrm{supp}(\widehat{\psi_j}) \subset P_j$. Let $v : J \to \mathbb{R}^+$, and $0 < p, q < \infty$. We define the associated decomposition space (quasi-)norm as

$$\|f\|_{\mathcal{P};v,p,q} = \left\| \left(v(j) \|f * \psi_j\|_{L^p} \right)_{j \in J} \right\|_{\ell^p} \in \mathbb{R}_0^+ \cup \{\infty\} ,$$

for $f \in \mathrm{L}^2(\mathbb{R}^d)$. We let $D(\mathcal{P}; v, p, q)$ denote the completion of

$$\{ f \in \mathrm{L}^2(\mathbb{R}^d) : \|f\|_{\mathcal{P};v,p,q} < \infty \} .$$

Informal description

Central message

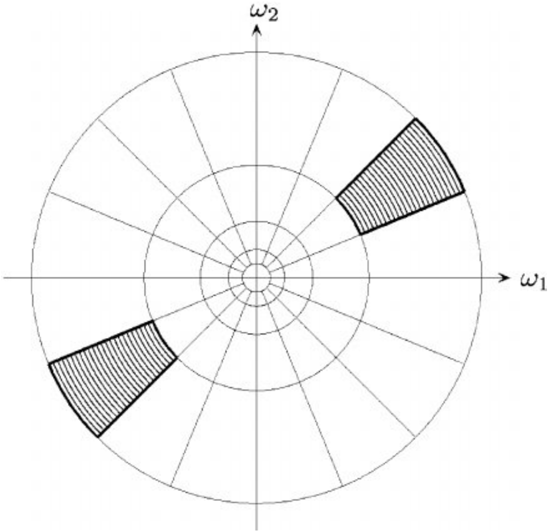
Decomposition spaces are independent of the choice of Ψ, under suitable assumptions.
I.e., the frequency covering determines the properties of the spaces.
(Pertinent notion: Structured admissible covering)

- $\|f * \psi_j\|_{L^p}$ measures size of the channel contribution.
- Depending on the choice of v, p, q: The Besov-type norm $\|f\|_{\mathcal{P};v,p,q}$ quantifies decay of channel contributions.
- Can be used to quantify smoothness of f: E.g., L^2-Sobolev spaces have a description as Besov-type spaces.
- In higher dimensions: Using oriented frequency bands allows to quantify directional notions of smoothness.

Example classes

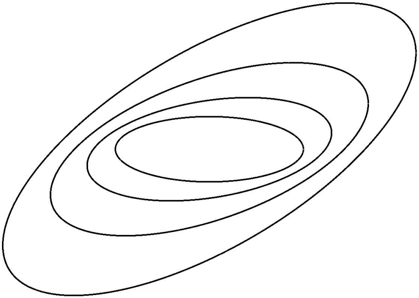
- $\Psi = (\psi_j)_{j\in\mathbb{Z}}$ a wavelet system, $0 < p, q \leq \infty$, $v(j) = 2^{js}$.
  $B(\mathcal{P}; v, p, q) = \dot{B}^s_{p,q}$, the homogeneous Besov space. (Besov, Triebel, etc.)
- $\Psi = (\psi_j)_{j\in\mathbb{Z}}$ a Gabor system, $0, p, q \leq \infty$, $v(j) = (1 - |j|)^\alpha$.
  $B(\mathcal{P}; v, p, q) = M^\alpha_{p,q}$, a modulation space (Feichtinger et.al.)
- $\Psi = (\psi_j)_{j\in\mathbb{Z}}$ an anisotropic wavelet system associated to the matrix A.
  $D(\mathcal{P}; v, p, q) = \dot{B}^s_{p,q}(A)$, the homogeneous anisotropic Besov space.
- Let Ψ denote a shearlet system, v a suitable weight.
  $D(\mathcal{P}; v, p, q) = Co_{\mathcal{SH};p,q,v}$, the shearlet coorbit space.
  (Dahlke/Kutyniok/Steidl/Teschke)
- Further (huge) classes: α-modulation spaces, inhomogeneous anisotropic Besov spaces, shearlet smoothness spaces, curvelet smoothness spaces, generalized wavelet coorbit spaces, ...

Oriented wavelet covering



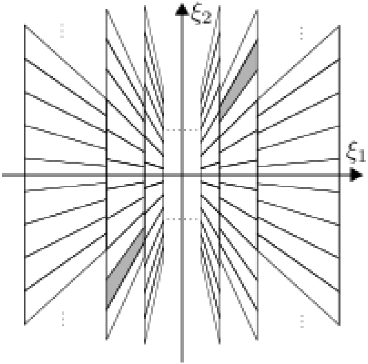
Corresponds to (isotropic) Besov spaces

Anisotropic covering



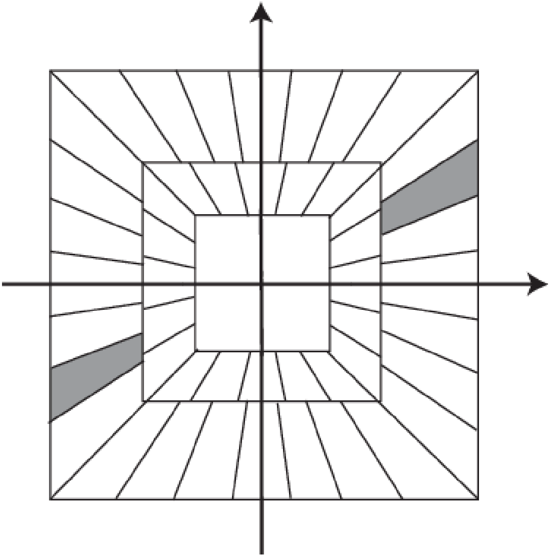
Corresponds to anisotropic Besov spaces.

Shearlet covering



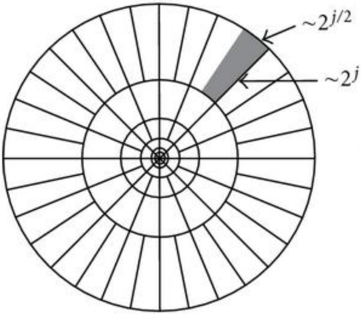
Corresponds to shearlet coorbit spaces

Cone-adapted shearlet covering



Corresponds to shearlet smoothness spaces. (Labate et.al.)

Curvelet covering



Corresponds to curvelet smoothness spaces

Understanding decomposition spaces

Challenges

- Extract properties of the decomposition spaces associated to a covering $\mathcal{P}$ directly from the covering.
- Classification: Give criteria for equivalence of coverings, defined by the fact that the induced scale of decomposition spaces coincides. (Main topic today.)
- How are these spaces related to classical smoothness spaces such as Besov or Sobolev spaces?

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A metric classification criterion

Lemma (Feichtinger/Gröbner)

Let $\mathcal{Q} = (Q_i)_{i\in I}$ denote a structured admissible covering. Then $\mathcal{Q}$ induces a metric $d_{\mathcal{Q}}$ on $\mathcal{O} = \bigcup_i Q_i$ via

$$\begin{aligned} d_{\mathcal{Q}}(x,y) &= \min\{n \in \mathbb{N} : \exists x = x_0, x_1, \ldots, x_n = y, \\ &\quad \forall j = 1, \ldots, n \exists i \in I : \{x_{j-1}, x_j\} \subset Q_i\} \end{aligned}$$

for $x \neq y$.

Theorem (Feichtinger/Gröbner, F. Voigtlaender, R. Koch)

Let $\mathcal{P} = (P_i)_{i\in I}, \mathcal{Q} = (Q_j)_{j\in J}$ denote structured admissible coverings, consisting of connected sets, covering the same frequency set $\mathcal{O}$. Then $\mathcal{Q}$ and $\mathcal{P}$ are equivalent iff $\mathrm{id} : (\mathcal{O}, d_{\mathcal{P}}) \to (\mathcal{O}, d_{\mathcal{Q}})$ is a quasi-isometry, i.e., there exist $a, A > 0$ and $b, B \geq 0$ such that

$$\forall x, y \in \mathcal{O} \ : a d_{\mathcal{Q}}(x,y) - b \leq d_{\mathcal{P}}(x,y) \leq A d_{\mathcal{Q}} + B \ .$$

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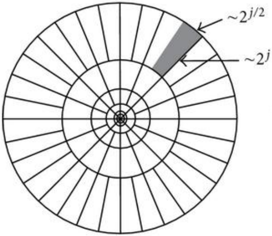
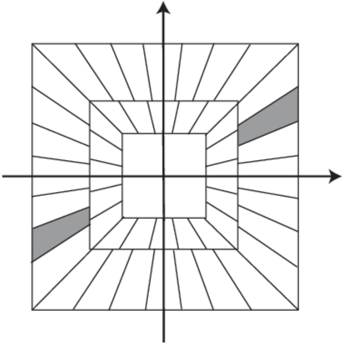
Application I: Curvelet vs. Shearlet smoothness spaces





Labate et.al.: Curvelet and Shearlet smoothness spaces coincide!

Shearlet dilation groups

(i) Original shearlet dilation group(s) (Dahlke/Kutyniok/Maass/Sagiv/Teschke):

$$H = \left\{ \begin{pmatrix} a & s_1 & \ldots & s_{d-1} \\ & a^{\alpha_2} & & \\ & & \ddots & \\ & & & a^{\alpha_d} \end{pmatrix} : a > 0, s_1, \ldots, s_{d-1} \in \mathbb{R} \right\} .$$

$\alpha_2, \ldots, \alpha_d$ suitably chosen.

(ii) Toeplitz shearing subgroup (Dahlke,Teschke, Häuser)

$$H = \left\{ \begin{pmatrix} a & s_1 & s_2 & \ldots & \ldots & s_{d-1} \\ & a & s_1 & s_2 & \ldots & s_{d-2} \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & \ddots & s_2 \\ & & & & & s_1 \\ & & & & & a \end{pmatrix} : a > 0, s_1, \ldots, s_{d-1} \in \mathbb{R} \right\} .$$

General shearlet dilation groups

Definition

Let $H < \mathrm{GL}(d, \mathbb{R})$ denote an irreducibly admissible dilation group. H is called generalized shearlet dilation group, if there exist two closed subgroups $S, D < H$ with the following properties:

(i) S is a connected closed abelian matrix group consisting of upper triangular matrices with ones on the diagonal.

(ii) $D = \{\exp(rY) : r \in \mathbb{R}\}$ is a one-parameter group, where Y is a diagonal matrix.

(iii) Every $h \in H$ can be written uniquely as $h = \pm ds$, with $d \in D$ and $s \in S$.

S is called the shearing subgroup of H, and D the scaling subgroup.

Shearlet transform

- A shearlet dilation group H defines a (continuously indexed) generalized wavelet system $\psi_h(x) = |\det(h)|^{-1/2}\psi(h^{-1}x)$.
- Associated shearlet transform:

$$\mathcal{SH}_H f(x,h) = f * \psi_h^*(x) \;, x \in \mathbb{R}^n, h \in H$$

- Shearlet coorbit spaces $Co_{\mathcal{SH}_H}$ are defined by norms measuring shearlet coefficient decay.
- These spaces have a decomposition space characterization, based on a covering of the open dual orbit $H^T(1,0,\ldots,0)^T = \mathbb{R}^* \times \mathbb{R}^{d-1}$. Let

$$p_H : h \mapsto h^T(1,0,\ldots,0)^T \;.$$

- Question: Which shearlet dilation groups have the same scale of coorbit spaces?
  (Answer is encoded in the dual actions.)

Abelian shearlet dilation groups

Classification up to conjugacy

- If $H < \mathrm{GL}(d, \mathbb{R})$ is an abelian shearlet dilation group, its Lie algebra $\mathfrak{h}$ fulfills
$$\mathfrak{h} = \mathrm{span}(H) \ .$$
- In particular, $\mathfrak{h}$ is a d-dimensional associative algebra of the type $\mathfrak{h} = \mathbb{R} \cdot 1 \oplus \mathcal{N}$, and $\mathcal{N}$ consists of strictly upper triangular matrices.
- The conjugacy classes of abelian shearlet dilation groups are in bijective correspondence to the isomorphy classes of associative commutative algebras of the type $\mathcal{A} = \mathbb{R} \cdot 1_{\mathcal{A}} \oplus \mathcal{N}$, with $\mathcal{N}$ nilpotent.
- From dimension 7 on, there are uncountably many such isomorphism classes.

Classifying abelian shearlet dilation groups

Theorem (R. Koch,HF)

Let H_1 and H_2 be two abelian shearlet dilation groups with dual orbit $\mathcal{O} = \mathbb{R}^* \times \mathbb{R}^{d-1}$. Then the following are equivalent:

(a) *H_1 and H_2 have the same coorbit spaces.*

(b) *If p_{H_1}, p_{H_2} denote the associated dual orbit maps, then $p_{H_2}^{-1} \circ p_{H_1} : H_1 \to H_2$ is quasi-isometric with respect to the word metrics on the groups.*

(c) *There exists an invertible matrix C fulfilling $C^T \mathcal{O} = \mathcal{O}$ as well as $C^{-1} H_1 C = H_2$.*

Sketch of proof

- $(a) \Leftrightarrow (b)$ and $(c) \Rightarrow (a)$ hold for more general dilation groups.
- $(b) \Rightarrow (c)$, Step 1: Let $\mathfrak{h}_i$ denote the associated Lie algebras. If H_i is an abelian shearlet dilation group, $\mathfrak{h}_i$ is an associative matrix algebra.
- $(b) \Rightarrow (c)$, Step 2: $\exp : (\mathfrak{h}_i, +) \to H_i$ is a group isomorphism onto the connected component of H_i.
- $(b) \Rightarrow (c)$, Step 3: As a consequence,

$$\log \circ p_{H_2}^{-1} \circ p_{H_1} \circ \exp : \mathfrak{h}_1 \to \mathfrak{h}_2$$

  is a polynomial quasi-isometry (w.r.t. arbitrary norms), hence affine.
- $(b) \Rightarrow (c)$, Step 4: It follows that $p_{H_2}^{-1} \circ p_{H_1}$ is a group isomorphism, and $\mathfrak{h}_1, \mathfrak{h}_2$ are isomorphic associative algebras.
- $(b) \Rightarrow (c)$, Step 5: The algebra isomorphism gives rise to conjugacy.

Concluding remarks

- Central message: Decomposition space description allows unified treatment of a large variety of function spaces.
- The problem of classification is equivalent to quasi-isometry of induced metrics.
- The method can also be employed to classify anisotropic Besov spaces.
- There is a rich and far-reaching embedding theory for decomposition spaces, into classical smoothness spaces, and into other decomposition spaces.
- Sample applications: Explicit conditions for
  - $\mathcal{D}(\mathcal{P}, \mathrm{L}^p, \ell^q_v) \hookrightarrow W^p_k$,
  - $\mathcal{D}(\mathcal{P}, \mathrm{L}^p, \ell^q_v) \hookrightarrow \dot{B}^s_{p,q}$
  - $\mathcal{D}(\mathcal{P}, \mathrm{L}^p, \ell^q_v) \hookrightarrow \mathrm{L}^p(\mathbb{R}^d)$.
- The embedding results are quite technical to apply, and their understanding could possibly profit from metric reformulation

Thank you !