Riesz transforms on a class of non-doubling manifolds

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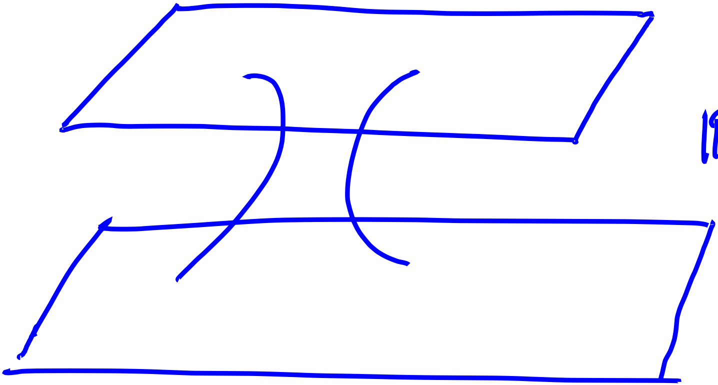
Abstract

Abstract: We consider a class of manifolds $\mathcal{M}$ obtained by taking the connected sum of a finite number of N-dimensional Riemannian manifolds of the form $(\mathbb{R}^{n_i}, \delta) \times (\mathcal{M}_i, g)$, where $\mathcal{M}_i$ is a compact manifold, with the product metric. The case of greatest interest is when the Euclidean dimensions n_i are not all equal. This means that the ends have different 'asymptotic dimension', and implies that the Riemannian manifold $\mathcal{M}$ is not a doubling space. We completely describe the range of exponents p for which the Riesz transform on $\mathcal{M}$ is a bounded operator on $L^p(\mathcal{M})$. Namely, under the assumption that each n_i is at least 3, we show that Riesz transform is of weak type $(1,1)$, is continuous on L^p for all

$$1 < p < \min_i n_i$$

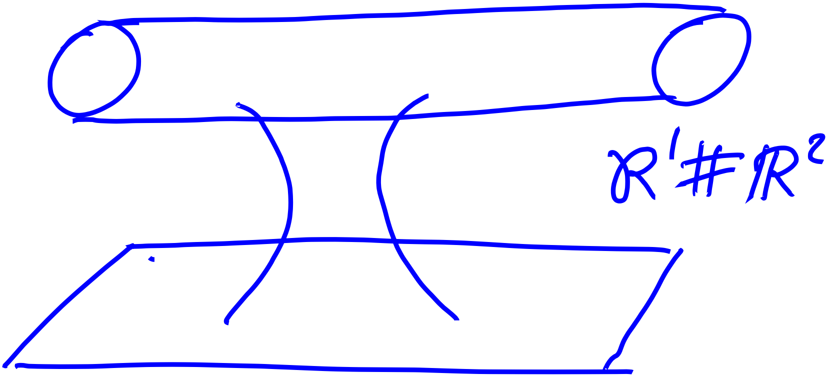
and is unbounded on L^p otherwise.
When $\min_i n_i = 2$ then the boundedness range is the interval $(1, 2]$.

Manifolds with ends



$\mathbb{R}^2 \# \mathbb{R}^2$

Manifolds with ends



If the dimensions at infinity for ends are not equal the doubling condition fails.

Riesz transform - Definitions

$\mathcal{M}$ complete Riemannian manifold and $d = \nabla$ the corresponding gradient.

Δ - Laplace-Beltrami operator defined by the following quadratic form

$$\langle \Delta f, g \rangle = \int_M g \Delta f d\mu = \int \nabla f \cdot \nabla g \, d\mu.$$

Riesz transform - Definitions

Riesz transform on is bounded on L^p:

$$(R_p) \qquad \| |\nabla f| \|_p \le C \|\Delta^{1/2} f\|_p, \ \forall f \in L^p(X, \mu)$$

Note that $\| |\nabla f| \|_2 = \|\Delta^{1/2} f\|_2$ so $(R_p 2)$ holds automatically.
Essentially we ask for the range of p such that $\nabla \Delta^{-1/2}$ is bounded on L^p

Doubling condition

Let us recall that a metric measured space $(\mathcal{X}, d, \mu)$ with metric d and Borel measure μ is said to satisfy the *doubling condition* that is if there exists universal constant C such that

$$(D) \qquad \mu(B(x, 2r)) \leq C\mu(B(x, r)) \quad \forall\, r > 0,\ x \in X.$$

Here by $B(x, r)$ we denote the ball of radius r centred at $x \in X$.

In the setting we consider here the doubling condition fails
Also No Poincaré, No Harnack

Riesz transform

Riesz transform - classical topic - M. Riesz around century ago (1928).

A) Significance: Definition of Sobolev spaces, theory of PDE - what does it mean to solve PDE.

B) Calderon-Zygmund nature of Riesz transform - Hence the area really took off somewhere in ’70.

C) Natural generalisation, Manifolds, Schrödinger, Dirichlet forms, Higher orders, etc.

Manifolds with ends

Grigor'yan and Saloff-Coste (re)introduced a notion of manifolds with ends and obtained satisfactory estimates for the corresponding heat kernels via probabilistic methods.

Riesz transform

T. Coulhon, X.T. Duong, Riesz transforms for $1 \leq p \leq 2$ is bounded under the doubling condition with Gaussian bounds for the corresponding heat kernel. The Riesz transform problem is not symmetric. The doubling condition assumption seems to be essential, does not apply to the setting which we consider here.

G. Carron, Th. Coulhon and A. Hassell Manifolds - two connected copies of R^n (R_p) hods if and only if $1 < p < n$ - (end points not discussed)

G. Carron (R_p) holds if $\frac{n}{n-1} < p < n$ where $n = \min_i n_i$.

Our result I

Hassell-S.

Theorem

Suppose that $\mathcal{M} = (\mathbb{R}^{n_1} \times \mathcal{M}_1)\# \dots \#(\mathbb{R}^{n_l} \times \mathcal{M}_l)$ *is a manifold with* $l \geq 2$ *ends , with* $n_i \geq 3$ *for each* i*. Then the Riesz transform* $\nabla\Delta^{-1/2}$ *defined on* $\mathcal{M}$ *is bounded on* $L^p(\mathcal{M})$ *if and only if* $1 < p < \min\{n_1, \cdots, n_l\}$*. That is, there exists* C *such that*

$$\Big\| |\nabla\Delta^{-1/2} f| \Big\|_p \leq C\|f\|_p, \ \forall f \in L^p(X, \mu)$$

if and only if $1 < p < \min\{n_1, \cdots, n_l\}$*. In addition the Riesz transform* $\nabla\Delta^{-1/2}$ *is of weak type* $(1, 1)$*.*

Testing ideas - Riesz transform in one dimension

For $d > 1$ consider the space $L^2(\mathbb{R}, (1+|r|)^{d-1}dr)$ and the operator $L = \nabla^*\nabla$, where $\nabla f = f'$ is the derivative operator and ∇^* is the adjoint with respect to the measure $(1+|r|)^{d-1}$. Then we have

Theorem

Let L be as above. The Riesz transform $dL^{-1/2}$ is bounded on $L^p(\mathbb{R}, (1+|r|)^{d-1}dr)$ if and only if

(i) $1 < p < d$ *for* $d > 2$

(ii) $1 < p \leq 2$ *for* $d = 2$

(iii) $1 < p < \frac{d}{d-1}$ *for* $1 < d < 2$.

The main point of the proof is to construct the resolvent of the Laplace operator $(\Delta + \lambda^2)^{-1}$. A good intuition can be obtain from considering one dimension operator

$$(\Delta_{(m,n)}f)(r) = \begin{cases} -\frac{d}{dr^2} - \frac{n-1}{r}\frac{d}{dr}f(r) & r > 1 \\ -\frac{d}{dr^2} - \frac{m-1}{r}\frac{d}{dr}f(r) & r < -1. \end{cases} \tag{1}$$

The point -1 is identified with 1.

Now we note that any solution of the equation

$$f'' + \frac{n-1}{r} f' = \lambda^2 f. \tag{2}$$

is a linear combination of the functions $r \to I_n(\lambda r)$ and $r \to k_n(\lambda r)$, where

$$I_n(r) = r^{1-n/2} I_{n/2-1}(r) \quad \text{and} \quad k_n(r) = r^{1-n/2} K_{|n/2-1|}(r).$$

where modified Bessel functions are modified Bessel functions.

Riesz transform in 1-D kernel of the resolvent

A part of the whole formula for the kernel.

$$\mathbf{Q}_\lambda(z,z') = \begin{cases} \frac{1}{\lambda}\dfrac{k_n(\lambda z)k_m(-\lambda x)}{(k_n k_m' + k_n' k_m)(\lambda)} & x \le -1, 1 \le y \\ \frac{1}{\lambda}\dfrac{k_n(\lambda x)k_m(-\lambda y)}{(k_n k_m' + k_n' k_m)(\lambda)} & y \le -1, 1 \le x \\ 0 & \textit{otherwise}. \end{cases} \tag{3}$$

Main point of the proof

Let's go back to the Manifolds with ends setting
One should try to understand the corresponding resolvent rather then the heat kernel to study the Riesz transform via the formula

$$\nabla \Delta^{-1/2} = \frac{2}{\pi} \nabla \int_0^\infty (\Delta + k^2)^{-1} \, dk. \tag{4}$$

Key lemma

Lemma

Assume that each n_i is at least 3. Let $v \in C_c^\infty(\mathcal{M};\mathbb{R})$. Then there is a function $u\colon \mathcal{M}\times\mathbb{R}_+ \to \mathbb{R}$ such that $(\Delta + k^2)u = v$ and such that, on the ith end we have:

$$|u(z,k)| \le C\langle d(z_i^\circ, z)\rangle^{-(n_i-2)} \exp(-kd(z_i^\circ, z)) \quad \forall z \in \mathbb{R}^{n_i} \times \mathcal{M}_i$$
$$|\nabla u(z,k)| \le C\langle d(z_i^\circ, z)\rangle^{-(n_i-1)} \exp(-kd(z_i^\circ, z)) \quad \forall z \in \mathbb{R}^{n_i} \times \mathcal{M}_i \tag{5}$$

for some $c, C > 0$.

Low Energy Resolvent Parametrix G_3 term

$$(\Delta + k^2)^{-1}(z, z') = G_1(k) + G_2(k) + G_1(k)$$

$$\begin{aligned} G_1(k) &= \sum_{i=1}^{l} (\Delta_{\mathbb{R}^{n_i} \times \mathcal{M}_i} + k^2)^{-1}(z, z')\phi_i(z)\phi_i(z') \\ G_2(k) &= G_{int}(k)\Big(1 - \sum_{i=1}^{l} \phi_i(z)\phi_i(z')\Big) \\ G_3(k) &= \sum_{i=1}^{l} (\Delta_{\mathbb{R}^{n_i} \times \mathcal{M}_i} + k^2)^{-1}(z_i^{\circ}, z')u_i(z, k)\phi_i(z'). \end{aligned} \tag{6}$$

G_1 is the resolvent acting on separate ends.
G_2 is the resolvent acting on the compact connecting part.
G_3 the interaction "perturbation" part.

Significance of the G_3 term

The range of p for which the Riesz transform is bounded on L^p is governed by the asymptotics of $G_3(k)$ part.

$G_3(k)$ is easy to understand because it (essentially)is of rank one that is

$$G_3(k)(z, z') = f_k(z)g_k(z)$$

for some function f_k and g_k.

Our result II

Hassell-Nix-S.

Theorem

Suppose that $\mathcal{M} = (\mathbb{R}^2 \times \mathcal{M}_1)\#(\mathbb{R}^n \times \mathcal{M}_2)$ *is a manifold with* 2 *ends with* $n \geq 3$. *Then the Riesz transform* $\nabla\Delta^{-1/2}$ *defined on* $\mathcal{M}$ *is bounded on* $L^p(\mathcal{M})$ *if and only if* $1 < p \leq 2$. *In addition the Riesz transform* $\nabla\Delta^{-1/2}$ *is of weak type* $(1, 1)$.

Doubling condition and Riesz transform

Some results relevant to boundedness of Riesz transform without the doubling condition.
Dimension free estimates - Stein also Coulhon, Müller, Zienkiwicz

Hebisch and Steger (see also Martini, Ottazzi, Vallarino) proved the boundedness of the Riesz transform on all L^p spaces $1 < p < \infty$ for a class of Laplace operators acting on some Lie groups of exponential growth where of course the doubling condition fails.

Open problem I

L considered Laplace Beltrami operator and F nice compactly supported function F supp $F \subset (a, b)$, where $0 < a < b$. Then

$$\sup_{t>0} \|F(t\Delta)\|_{p\to p} \le C?$$

Interesting example (forget support)

$$\|\exp(-t(1+i)\Delta)\|_{p\to p} \le C?$$

Open problem II

Consider

$$L = \partial_x^2 + \partial_y^2 g(x)$$

acting on $\mathbb{R}^2$ where $g(x) = x^2$ for $x > 0$ and $g(x) = x^4$ for $x < 0$. Prove the the corresponding Riesz transform is bounded on L^p for all $1 < p \leq 2$.

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