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Calderón's type reproducing formula related to the q-Dunkl two wavelet theory

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Abstract

In this paper, using some elements of the q-harmonic analysis associated to the q-Dunkl operator introduced by N. Bettaibi et al. in [1], for fixed $0<q<1$, the notion of a q-Dunkl two-wavelet is introduced. The resolution of the identity formula for the q-Dunkl continuous wavelet transform is then formulated and proved. Calderón's type reproducing formula in the context of the q-Dunkl two wavelet theory is proved.

Introduction

Calderón formula [3] involving convolution related to the Fourier transform is useful in obtaining reconstruction formula for wavelet transform besides many other applications in decomposition of certain function spaces. It is expressed as follows:

$$\Phi(\xi)=\frac{1}{c_{\varphi,\phi}}\int_0^{+\infty}\Phi*\varphi_t*\phi_t(\xi)\frac{dt}{t},\quad \xi\in\mathbb{R},\qquad (1)$$

where

$$\varphi_t(x):=\frac{1}{t}\varphi\left(\frac{x}{t}\right),\quad \phi_t(x):=\frac{1}{t}\phi\left(\frac{x}{t}\right),\quad \forall x\in\mathbb{R},$$

$c_{\varphi,\phi}$ is a constant depending on functions φ,ϕ and $*$ denotes a convolution operation.

The aim of this paper is to give a q-version of Calderón's type reproducing formula (1) in the context of q-Dunkl harmonic analysis, more precisely, we study similar question when in (1), the classical convolution $*$ is replaced by a generalized q-Dunkl convolution $*_{q,\alpha}$ on the real line generated by the q-Dunkl differential operator $\Lambda_{q,\alpha}$, $\alpha\geq -1/2$.

Preliminaries and q-Notations

Throughout this paper we assume that $\alpha\geq-1/2$ and $0<q<1$.

1. We denote

$$\mathbb{R}_q=\{\pm q^n,\ n\in\mathbb{Z}\},\quad \mathbb{R}_q^+=\{q^n,\ n\in\mathbb{Z}\}\quad\text{and}\quad \widehat{\mathbb{R}_q}=\mathbb{R}_q\cup\{0\}.$$

2. For complex number a, the q-shifted factorials are defined by:

$$(a;q)_0=1,\quad (a;q)_n=\prod_{l=0}^{n-1}(1-aq^l),\ n=1,2,...,\quad (a;q)_\infty=\prod_{l=0}^{\infty}(1-aq^l).$$

and we also denote for all $x\in\mathbb{C}$ and $n\in\mathbb{N}$

$$[x]_q=\frac{1-q^x}{1-q},\qquad [n]_q!=[1]_q\times[2]_q...\times[n]_q=\frac{(q;q)_n}{(1-q)^n}.$$

3. The q-Jackson integrals from 0 to a and from $-\infty$ to $+\infty$ are defined by

$$\int_0^a f(x)d_qx=(1-q)a\sum_{n=0}^{+\infty}q^nf(aq^n),$$

$$\int_{-\infty}^{+\infty}f(x)d_qx=(1-q)\sum_{n=-\infty}^{+\infty}q^n\left[f(q^n)+f(-q^n)\right].$$

4. We denote by $L^p_{q,\alpha}(\mathbb{R}_q)$, $p\in[1,+\infty]$, the set of all real functions on $\mathbb{R}_q$ for which

$$\|f\|_{q,p,\alpha}=\begin{cases}\left(\int_{-\infty}^{+\infty}|f(x)|^p|x|^{2\alpha+1}d_qx\right)^{1/p}<+\infty & \text{if } 1\leq p<+\infty,\\ ess\sup_{x\in\mathbb{R}_q}|f(x)|<+\infty & \text{if } p=+\infty,\end{cases}$$

5. The q-Dunkl translation operator is defined for $f\in L^2_{q,\alpha}(\mathbb{R}_q)$ and $x,y\in\mathbb{R}_q$ by

$$\mathcal{T}^{q,\alpha}_x(f)(y)=c_{q,\alpha}\int_{-\infty}^{+\infty}\mathcal{F}^{q,\alpha}_D(f)(\lambda)\Psi_{q,\alpha}(\lambda x)\Psi_{q,\alpha}(\lambda y)|\lambda|^{2\alpha+1}d_q\lambda.$$

The q-Dunkl translation operators allow us to define a q-Dunkl convolution product $*_{q,\alpha}$ as follows: for all $f,g\in\mathcal{S}_q(\mathbb{R}_q)$, we have

$$f*_{q,\alpha}g(x)=c_{q,\alpha}\int_{-\infty}^{\infty}T^{q,\alpha}_x(f)(-y)g(y)|y|^{2\alpha+1}d_qy,\quad \forall x,y\in\mathbb{R}_q,$$

provided the q-integral exists.

q-Dunkl Harmonic analysis associated with $\Lambda_{q,\alpha}$

1. The q-Dunkl operator $\Lambda_{q,\alpha}$ is defined by

$$\Lambda_{q,\alpha}(f)(x)=\partial_q[\mathcal{H}_{q,\alpha}(f)](x)+[2\alpha+1]_q\frac{f(x)-f(-x)}{2x},$$

where

$$\mathcal{H}_{q,\alpha}:f=f_e+f_o\mapsto f_e+q^{2\alpha+1}f_o,$$

with f_e and f_o are respectively, the even and the odd parts of f.
The q-Dunkl kernel is defined on $\mathbb{R}_q$ by

$$\Psi_{q,\alpha}(z)=j_\alpha(z,q^2)+\frac{iz}{[2\alpha+2]_q}j_{\alpha+1}(z,q^2),\quad z\in\mathbb{R}_q,$$

where $j_\alpha(.,q^2)$ is the normalized third Jackson's q-Bessel function given by

$$j_\alpha(x,q^2)=\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma_{q^2}(\alpha+1)q^{n(n+1)}}{\Gamma_{q^2}(\alpha+n+1)\Gamma_{q^2}(n+1)}\left(\frac{x}{1+q}\right)^{2n}.$$

2. the function $\Psi_{q,\alpha}(\lambda.)$, $\lambda\in\mathbb{C}$ is the unique analytic solution of the q-differential-difference equation:

$$\begin{cases}\Lambda_{q,\alpha}(f)=i\lambda f,\\ f(0)=1.\end{cases}$$

3. The q-Dunkl transform $\mathcal{F}^{q,\alpha}_D$ is defined on $L^1_{q,\alpha}(\mathbb{R}_q)$ by

$$\mathcal{F}^{q,\alpha}_D(f)(\lambda)=c_{q,\alpha}\int_{-\infty}^{\infty}f(x)\Psi_{q,\alpha}(-\lambda x)|x|^{2\alpha+1}d_qx,\quad\forall\lambda\in\mathbb{R}_q,$$

where

$$c_{q,\alpha}=\frac{(1+q)^{-\alpha}}{2\Gamma_{q^2}(\alpha+1)}.$$

4. The q-Dunkl transform $\mathcal{F}^{q,\alpha}_D$ is an isomorphism from $\mathcal{S}_q(\mathbb{R}_q)$ onto itself and extends uniquely to an isometric isomorphism on $L^2_{q,\alpha}(\mathbb{R}_q)$ with:

$$\|\mathcal{F}^{q,\alpha}_D(f)\|_{q,2,\alpha}=\|f\|_{q,2,\alpha}.$$

5. If $f\in L^1_{q,\alpha}(\mathbb{R}_q)$ such that $\mathcal{F}^{q,\alpha}_D(f)\in L^1_{q,\alpha}(\mathbb{R}_q)$, then the q-inversion formula holds and we have

$$f(x)=\int_{-\infty}^{+\infty}\mathcal{F}^{q,\alpha}_D(f)(\lambda)\Psi_{q,\alpha}(\lambda x)d\mu_{q,\alpha}(\lambda).$$

q-Dunkl two-wavelet theory

Definition 1 (q-Dunkl wavelet)
A q-Dunkl wavelet is a square q-integrable function h on $\mathbb{R}_q$ satisfying the following admissibility condition

$$0<\mathcal{C}_{\alpha,h}=\int_0^{+\infty}|\mathcal{F}^{q,\alpha}_D(h)(a)|^2\frac{d_qa}{a}=\int_0^{+\infty}|\mathcal{F}^{q,\alpha}_D(h)(-a)|^2\frac{d_qa}{a}<\infty.$$

Definition 2 (q-Dunkl two-wavelet)
Let u and v be in $L^2_{q,\alpha}(\mathbb{R}_q)$. We say that the pair (u,v) is a q-Dunkl two-wavelet on $\mathbb{R}_q$ if for almost all $\lambda\in\mathbb{R}_q$, we have

$$\mathcal{C}_{\alpha,u,v}=\int_0^{+\infty}\mathcal{F}^{q,\alpha}_D(v)(a\lambda)\overline{\mathcal{F}^{q,\alpha}_D(u)(a\lambda)}\frac{d_qa}{a}<\infty.$$

Definition 3 (The continuous q-wavelet transform)
Let h be a q-Dunkl wavelet on $\mathbb{R}_q$ in $L^2_{q,\alpha}(\mathbb{R}_q)$. We define the continuous q-wavelet transform associated with the q-Dunkl operator for all $f\in L^2_{q,\alpha}(\mathbb{R}_q)$ by

$$\psi^{\alpha,D}_{q,h}(f)(a,x)=c_{q,\alpha}\int_{-\infty}^{+\infty}f(\lambda)\overline{h_{a,x}(\lambda)}|\lambda|^{2\alpha+1}d_q\lambda,$$

where

$$h_{a,x}(\lambda)=a^{\alpha+1}T^{q,\alpha}_x(h_a)(\lambda)\ \text{and}\ h_a(x)=\frac{1}{a^{2\alpha+2}}h\left(\frac{x}{a}\right),\quad\forall x\in\mathbb{R}_q.$$

Theorem 1 (Parseval formula)
Let (u,v) be a q-Dunkl two-wavelet. Then for all f and g in $L^2_{q,\alpha}(\mathbb{R}_q)$, there holds

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}\psi^{\alpha,D}_{q,u}(f)(a,x)\overline{\psi^{\alpha,D}_{q,v}(g)(a,x)}d\mu_{q,\alpha}(a,x)$$
$$=\mathcal{C}_{\alpha,u,v}\int_{-\infty}^{+\infty}f(x)\overline{g(x)}|x|^{2\alpha+1}d_qx.$$

Theorem 3 (Inversion formula) Let (u,v) be a q-Dunkl two-wavelet. For all f in $L^1_{q,\alpha}(\mathbb{R}_q)$ such that $\mathcal{F}^{q,\alpha}_D(f)$ belongs to $L^1_{q,\alpha}(\mathbb{R}_q)$, we have

$$f(\lambda)=\frac{c_{q,\alpha}}{\mathcal{C}_{\alpha,u,v}}\int_0^{+\infty}\int_{-\infty}^{+\infty}\psi^{\alpha,D}_{q,u}(f)(a,x)v_{a,x}(\lambda)d\mu_{q,\alpha}(a,x),\quad\forall\lambda\in\mathbb{R}_q.$$

Calderón's type reproducing formula in the context of the q-Dunkl two-wavelet

Theorem 1 (q-Calderón's type formula) Let u and v be two q-Dunkl wavelets in $L^2_{q,\alpha}(\mathbb{R}_q)$ such that (u,v) is a q-Dunkl two-wavelet, $\mathcal{C}_{\alpha,u,v}\neq0$, and $\mathcal{F}^{q,\alpha}_D(u)$ and $\mathcal{F}^{q,\alpha}_D(v)$ both belong to $L^\infty_{q,\alpha}(\mathbb{R}_q)$. Then, for all f in $L^2_{q,\alpha}(\mathbb{R}_q)$ and $0<\varepsilon<\delta<\infty$, the function

$$f^{\varepsilon,\delta}(\lambda)=\frac{c_{q,\alpha}}{\mathcal{C}_{\alpha,u,v}}\int_\varepsilon^\delta\int_{-\infty}^{+\infty}\psi^{\alpha,D}_{q,u}(f)(a,x)v_{a,x}(\lambda)|x|^{2\alpha+1}d_qx\frac{d_qa}{a^{2\alpha+3}}$$

belongs to $L^2_{q,\alpha}(\mathbb{R}_q)$, and satisfies

$$\lim_{\varepsilon\to0,\,\delta\to\infty}\|f^{\varepsilon,\delta}-f\|_{q,2,\alpha}=0.$$

Conclusion

In this paper, using some new elements of q-harmonic analysis related to the q-Dunkl transform $\mathcal{F}^{q,\alpha}_D$ introduced by in [1], we define and study the q-Dunkl two wavelet and the continuous q-wavelet transform associated with this q-harmonic analysis. In addition to several properties, we establish a Plancherel formula and an inversion theorem for this transform. As applications, we prove a Calderón's type reproducing formula in the context of the q-Dunkl two wavelet theory.

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