An extension of the Bessel-Wright transform in the class of Boehmians

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Abstract

In this paper, we first construct a suitable Boehmian space on which the Bessel-Wright transform can be defined and some desired properties are obtained in the class of Boehmians. Some convergence results are also established.

Introduction and preliminaries

The space of Boehmians is constructed using an algebraic approach that utilizes convolution and approximate identities or delta sequences. If the construction is applied to a function space and the multiplication is interpreted as convolution, the construction yields a space of generalized functions. Those spaces provide a natural setting for extensions of the Bessel-Wright transform newly introduced by Fitouhi et al. [3]. We cite here, as briefly as possible, some facts about harmonic analysis related to the Bessel-Wright operator $\Delta_{\alpha,\beta}$. For more details we refer to [3].

We consider, on $(0,\infty)$ the difference differential operator indexed by two parameters α and β

$$\Delta_{\alpha,\beta}f(x)=\frac{d^2f}{dx^2}(x)+\frac{2(\alpha+\beta)+1}{x}\frac{df}{dx}(x)+\frac{4\alpha\beta}{x^2}[f(x)-f(0)]. \quad (0.1)$$

These operators are very important in pure mathematics and especially in special functions and harmonic analysis. The Bessel-Wright operator admits as eigenfunction with $-\lambda^2$ as eigenvalue the Bessel-Wright function

$$j_{(\alpha,\beta)}(\lambda x)=\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+1+n)\Gamma(\beta+1+n)}\left(\frac{\lambda x}{2}\right)^{2n},\ (\lambda\in\mathbb{C}),$$

which is even and symmetric in α and β and coincides when $\alpha=0$ or $\beta=0$ with the normalized Bessel function given by

$$j_\alpha(\lambda x)=\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)}\left(\frac{\lambda x}{2}\right)^{2n},\ (\lambda\in\mathbb{C}).$$

Let $L^p_\alpha=L^p_\alpha(0,\infty)$ denote the class of measurable functions f on $(0,\infty)$ for which $\|f\|^p_\alpha<\infty$, where

$$\|f\|^p_\alpha=\left(\int_0^\infty|f(x)|^pd\mu_\alpha(x)\right)^{\frac{1}{p}},\ \ if\ p<\infty,$$

$$\|f\|_{\infty,\alpha}=\|f\|_\infty=ess\ sup_{x\in(0,\infty)}|f(x)|,$$

and $d\mu_\alpha(x)=x^{2\alpha+1}dx$.

The Bessel-Wright transform for $f\in L^p_\alpha$ is defined by

$$\mathcal{F}_{(\alpha,\beta)}(f)(\lambda)=c_\alpha\int_0^\infty f(x)j_{\alpha,\beta}(\lambda x)d\mu_\alpha(x) \quad (0.2)$$

where $c_\alpha=\frac{1}{2^\alpha\Gamma(\alpha+1)}$

The following two definitions are needed for our results.

Definition 0.1. The Mellin-type convolution product of first kind is defined by:

$$f\times g(y)=\int_0^\infty f(yx^{-1})x^{-1}g(x)dx. \quad (0.3)$$

Definition 0.2. Let $\alpha>-\frac{1}{2}$ and $f,g\in L^1(0,\infty)$.Then we define the product $\otimes$ of f and g by the integral

$$f\otimes g(y)=\int_0^\infty f(yt)g(t)d\mu_\alpha(t), \quad (0.4)$$

By using (0.3) and (0.4), we get the following proposition:

Proposition 0.1. *Let f, g, and h be integrable functions in $L^1(0,\infty)$ and let $y>0$. Then*

$$f\otimes(g\times h)(y)=(f\otimes g)\otimes h(y)$$

Proposition 0.2. *The Bessel-Wright transform $\mathcal{F}_{(\alpha,\beta)}$is a bounded linear operator from L^1_α to $\mathcal{C}_0$.*

Generated Spaces of Boehmians

The class of Boehmians was introduced to generalize regular operators [2]. The minimal structure necessary for the abstract construction of Boehmian spaces consists of the following elements:

i. A topological vector space $\mathfrak{a}$

ii. A commutative semigroup $(\mathfrak{b},\bullet)$

iii. An operation $\star:\mathfrak{a}\times\mathfrak{b}\longrightarrow\mathfrak{a}$ such that, for each $x\in\mathfrak{a}$ and $s_1,s_2\in\mathfrak{b}$,

$$x\star(s_1\bullet s_2)=(x\star s_1)\star s_2.$$

iv. A collection $\Delta\subset\mathfrak{b}^{\mathbb{N}}$ such that:

a. if $x,y\in\mathfrak{a},(s_n)\in\Delta,x\bullet s_n=y\bullet s_n$ for all n, then $x=y$;

b. if $(s_n),(t_n)\in\Delta$, then $(s_n\bullet t_n)\in\Delta$.

The elements of Δ are called delta sequences. Denote by Q the set

$$Q=\{(x_n,s_n):x_n\in\mathfrak{a},(s_n)\in\Delta,x_n\star s_m=x_m\star s_n\forall m,n\in\mathbb{N}\}.$$

If $(x_n,s_n),(y_n,t_n)\in Q,x_n\star t_m=y_m\star s_n\forall m,n\in\mathbb{N}$, then we say that $(x_n,s_n)\sim(y_n,t_n)$. The relation $\sim$ is an equivalence relation in Q. The space of equivalence classes in Q is denoted by $\mathfrak{B}$. The elements of $\mathfrak{B}$ are called Boehmians.

Between $\mathfrak{a}$ and $\mathfrak{B}$, there is a canonical embedding expressed as

$$x\to\frac{x\star s_n}{s_n}.$$

The operation $\star$ is extended to $\mathfrak{B}\times\mathfrak{b}$ as follows:

If $\left[\frac{(f_n)}{(s_n)}\right]\in\mathfrak{B}$ and $\phi\in\mathfrak{b}$, then $\left[\frac{(f_n)}{(s_n)}\right]\star\phi=\left[\frac{(f_n)\star\phi}{s_n}\right]$.

We establish the following technical result.

Lemma 0.1. *Let $f\in L^1_\alpha(0,\infty)$ and $\psi\in D(0,\infty)$. Then*

$$\mathcal{F}_{(\alpha,\beta)}(f\times\psi)(\lambda)=(\mathcal{F}_{(\alpha,\beta)}f\otimes\psi)(\lambda).$$

The spaces generated here are the space $\mathfrak{B}_1=\mathfrak{B}_1(L^1_\alpha,(D,\times),\times,\Delta)$ and the space $\mathfrak{B}_2=\mathfrak{B}_2(L^1_\alpha,(D,\times),\otimes,\Delta)$. We denote by Δ, the set of delta sequences (δ_n) of $D(0,\infty)$ with the following properties:

$$\int_0^\infty\delta_n(x)dx=1, \quad (0.5)$$

$$\int_0^\infty|\delta_n(x)|dx<m, \quad (0.6)$$

where m is a positive real number

$$supp\ \delta_n(x)\to1,\ as\ n\to\infty. \quad (0.7)$$

Let us now establish that $\mathfrak{B}_1$ is a Boehmian space. We prefer to omit the proof for $\mathfrak{B}_2$ as its details are simlar.

Theorem 0.1. *Let $f\in L^1_\alpha(0,\infty)$, $\psi\in D(0,\infty)$ and $\alpha>-\frac{1}{2}$. Then $f\times\psi\in L^1_\alpha(0,\infty)$*

Theorem 0.2. *Let $f\in L^1_\alpha(0,\infty)$ and $\psi_1,\psi_2\in D(0,\infty)$, $\alpha>-\frac{1}{2}$.Then*

i. $f\times(\psi_1+\psi_2)=f\times\psi_1+f\times\psi_2$,

ii. $f\times(\psi_1\times\psi_2)=(f\times\psi_1)\times(\psi_2)$,

iii. $(\lambda f)\times\psi_1=\lambda(f\times\psi_1)=f\times(\lambda\psi_1),\lambda\in\mathbb{C}$

Theorem 0.3. *Let $f_n\to f\in L^1_\alpha(0,\infty)$ as $n\to\infty$ and let $\psi\in D(0,\infty)$, $\alpha>-\frac{1}{2}$. Then*

$$f_n\times\psi\to f\times\psi\ \ as\ \ n\to\infty$$

in $L^1_\alpha(0,\infty)$.

Theorem 0.4. *Let $f\in L^1_\alpha(0,\infty)$ and let $(\delta_n)\in\Delta,\alpha>-\frac{1}{2}$. Then*

$$f\times\delta_n\to f\ as\ n\to\infty$$

in $L^1_\alpha(0,\infty)$.

A sequence of Boehmians (ζ_n) in $\mathfrak{B}_1$ is said to be δ convergent to a Boehmian ζ in $\mathfrak{B}_1$ denoted by $\zeta_n\xrightarrow{\delta}\zeta$, if there exists a delta-sequence (δ_n) such that

$$(\zeta_n\times\delta_k),(\zeta\times\delta_k)\in L^1_\alpha\ \ \forall k,n\in\mathbb{N},$$

and

$$(\zeta_n\times\delta_k)\to(\zeta\times\delta_k)\ \ as\ n\to\infty,\ in\ L^1_\alpha,\ \forall k\in\mathbb{N}.$$

A sequence of Boehmians (ζ_n) in $\mathfrak{B}_1$ is said to be Δ convergent to a Boehmian ζ in $\mathfrak{B}_1$ denoted by $\zeta_n\xrightarrow{\Delta}\zeta$, if there exists a delta-sequence $(\delta_n)\in\Delta$ such that $(\zeta_n-\zeta)\times\delta_n\in L^1_\alpha\ \forall n\in\mathbb{N}$ and $(\zeta_n-\zeta)\times\delta_n\to0$ as $n\to\infty$ in L^1_α.

Similarly, the following theorems generate the Boehmian space $\mathfrak{B}_2$.

Theorem 0.5. *Let $f\in L^1_\alpha(0,\infty)$ and $\psi\in D(0,\infty)$. Then $f\otimes\psi\in L^1_\alpha(0,\infty)$.*

Theorem 0.6. *Let $f\in L^1_\alpha(0,\infty)$ and $\psi_1,\psi_2\in D(0,\infty)$. Then*

i. $f\otimes(\psi_1+\psi_2)=f\otimes\psi_1+f\otimes\psi_2$,

ii. $(\lambda f)\otimes\psi_1=\lambda(f\otimes\psi_1)=f\otimes(\lambda\psi_1),\ \lambda\in\mathbb{C}$.

Theorem 0.7. *For $f\in L^1_\alpha(0,\infty)$ and $\psi_1,\psi_2\in D(0,\infty)$, the following relation is true:*

$$f\otimes(\psi_1\times\psi_2)=(f\otimes\psi_1)\otimes\psi_2.$$

Theorem 0.8. *i. Let $f_n\to f$ in $L^1_\alpha(0,\infty)$ as $n\to\infty$ and let $\psi\in D(0,\infty)$. Then $f_n\otimes\psi\to f\otimes\psi$ as $n\to\infty$.*

ii. Let $f_n\to f$ in $L^1_\alpha(0,\infty)$ and let $(\delta_n)\in\Delta$.Then $f_n\otimes\delta_n\to f$ as $n\to\infty$.

The Bessel-Wright Transform of a Boehmian

Let $\zeta\in\mathfrak{B}_1$ and $\zeta=[\frac{(f_n)}{(\delta_n)}]$. Then, for every $\alpha>-\frac{1}{2}$ we define the generalized Bessel-Wright transform of ζ as follows:

$$\mathcal{F}^{ge}_{\alpha,\beta}\left(\left[\frac{(f_n)}{(\delta_n)}\right]\right)=\left[\frac{(\mathcal{F}_{(\alpha,\beta)}f_n)}{(\delta_n)}\right] \quad (0.8)$$

Theorem 0.9. *$\mathcal{F}^{ge}_{\alpha,\beta}$ is an isomorphism from $\mathfrak{B}_1$ into $\mathfrak{B}_2$.*

In addition, we now deduce the formula of extension of $\times$ to $\mathfrak{B}_1$ as follows:

$$\mathcal{F}^{ge}_{\alpha,\beta}\left(\left[\frac{(f_n)}{(\omega_n)}\right]\times\phi\right)=\mathcal{F}^{ge}_{\alpha,\beta}\left(\left[\frac{(f_n)}{(\omega_n)}\right]\right)\otimes\phi.$$

It can be proved as follows: By virtue of (0.8) we can write

$$\mathcal{F}^{ge}_{\alpha,\beta}\left(\left[\frac{(f_n)}{(\omega_n)}\right]\times\phi\right)=\left(\left[\frac{(\mathcal{F}^{ge}_{\alpha,\beta}(f_n\times\phi))}{(\omega_n)}\right]\right).$$

Hence, Lemma 0.1 gives

$$\mathcal{F}^{ge}_{\alpha,\beta}\left(\left[\frac{(f_n)}{(\omega_n)}\right]\times\phi\right)=\left(\left[\frac{(\mathcal{F}^{ge}_{\alpha,\beta}f_n\otimes\phi)}{(\omega_n)}\right]\right).$$

The definition of the product $\times$ implies that

$$\mathcal{F}^{ge}_{\alpha,\beta}\left(\left[\frac{(f_n)}{(\omega_n)}\right]\times\phi\right)=\left(\left[\frac{(\mathcal{F}^{ge}_{\alpha,\beta}f_n)}{(\omega_n)}\right]\right)\times\phi.$$

Thus, it follows from relation (0.8) that

$$\mathcal{F}^{ge}_{\alpha,\beta}\left(\left[\frac{(f_n)}{(\omega_n)}\right]\times\phi\right)=\mathcal{F}^{ge}_{\alpha,\beta}\left(\left[\frac{(f_n)}{(\omega_n)}\right]\right)\otimes\phi.$$

Hence, it is now possible to conclude that

$$\mathcal{F}^{ge}_{\alpha,\beta}\left(\left[\frac{(f_n)}{(\omega_n)}\right]\times\phi\right)=\mathcal{F}^{ge}_{\alpha,\beta}\left(\left[\frac{(f_n)}{(\omega_n)}\right]\right)\otimes\phi.$$

Theorem 0.10. *$\mathcal{F}^{ge}_{\alpha,\beta}:\mathfrak{B}_1\to\mathfrak{B}_2$ is continuous with respect to the $\delta-$convergence and $\Delta-$convergence.*

References

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Further informations

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