Geometric Hardy inequalities on starshaped sets

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Abstract

This work [2] is devoted to deriving the geometric Hardy inequalities on the starshaped sets in Carnot groups with some examples on Heisenberg and Engel groups. Also, the geometric Hardy inequalities are obtained on half-spaces for general vector fields, with the exmaple on the Grushin plane.

Preliminaries

Sub-Riemannian manifold: Let M be a smooth manifold of dimension n, with a given family of vector fields $\{X_k\}_{k=1}^N$, $n \geq N$, defined on M and satisfying the Hörmander condition. They induce a sub-Riemannian metric $\langle \cdot,\cdot\rangle_{\mathcal{H}}$ on the associated space $\mathcal{H}_x = \mathrm{span}(X_1(x),\dots,X_N(x))$. The triple $(M,\mathcal{H},\langle\cdot,\cdot\rangle_{\mathcal{H}})$ is a sub-Riemannian manifold. Note that, unlike for Carnot groups, in general, it is not possible to define dilations, translations, the homogeneous norm and the distance in the setting of general sub-Riemannian manifolds. Let us denote the operator of the sum of squares of vector fields by

$$\mathcal{L} := \sum_{k=1}^{N} X_k^2,$$

where $\nabla_X := (X_1,\dots,X_N)$.

Grushin plane: One of the important examples of a sub-Riemannian manifold is the Grushin plane. The Grushin plane is the space $\mathbb{R}^2$ with vector fields

$$X_1 = \frac{\partial}{\partial x_1}, \text{ and } X_2 = x_1\frac{\partial}{\partial x_2}.$$

Carnot groups: Let $\mathbb{G} = (\mathbb{R}^n, \circ, \delta_\lambda)$ be a stratified Lie group (or a homogeneous Carnot group or just a Carnot group), with the dilation structure δ_λ and Jacobian generators $X_1,\dots,X_N$, so that N is the dimension of the first stratum of $\mathbb{G}$. Let us denote by Q the homogeneous dimension of $\mathbb{G}$. We refer to the recent book [4] for extensive discussions of stratified (Carnot) Lie groups and their properties.

Heisenberg groups: The Heisenberg group is the most common example of a step 2 stratified group (Carnot group). The Lie algebra $\mathfrak{h}$ of the left-invariant vector fields on the Heisenberg group $\mathbb{H}_1$ is spanned by

$$X_1 := \frac{\partial}{\partial x_1} + 2x_2\frac{\partial}{\partial x_3},\quad X_2 := \frac{\partial}{\partial x_2} - 2x_1\frac{\partial}{\partial x_3},$$

with their (non-zero) commutator $[X_1,X_2] = -4\frac{\partial}{\partial x_3}$.

Engel groups: The Engel group is a well-known example of a step 3 stratified group (Carnot group). Let $\mathbb{E}$ be the Engel group, with the vector fields

$$X_1 := \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3} - \left(\frac{x_3}{2} + \frac{x_1x_2}{12}\right)\frac{\partial}{\partial x_4},$$

$$X_2 := \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3} + \frac{x_1^2}{12}\frac{\partial}{\partial x_4},$$

$$X_3 := [X_1,X_2] = \frac{\partial}{\partial x_3} + \frac{x_1}{2}\frac{\partial}{\partial x_4},\quad X_4 := [X_1,X_3] = \frac{\partial}{\partial x_4}.$$

Definition (Starshapedness [1])

Let $\Omega \subset \mathbb{G}$ be a C^1 domain containing the identity e. Then Ω is starshaped with respect to e if for every $x \in \partial\Omega$ one has

$$\langle Z(x), n(x)\rangle \geq 0, \tag{1}$$

where n is the Riemannian outer normal to $\partial\Omega$.
When the strict inequality holds, then Ω is said to be strictly starshaped with respect to e.
Here the vector fields Z are the infinitesimal generators of this group automorphism. This vector field Z takes the form

$$Z = \sum_{i=1}^{N} x_i'\frac{\partial}{\partial x_i'} + 2\sum_{l=1}^{N_2} x_{2,l}\frac{\partial}{\partial x_{2,l}} + \cdots + r\sum_{l=1}^{N_r} x_{r,l}\frac{\partial}{\partial x_{r,l}}. \tag{2}$$

Then for $x' \in \mathbb{R}^N$ and $x^{(i)} \in \mathbb{R}^{N_i}$ with $i = 2,\dots,r$, we have

$$\langle Z(x), n(x)\rangle = x'n' + 2x^{(2)}n^{(2)} + \ldots + rx^{(r)}n^{(r)}, \tag{3}$$

and $Z(x) = (x', 2x^{(2)}, \cdots, rx^{(r)})$, since $n(x) := (n', n^{(2)},\dots,n^{(r)})$ with $n' \in \mathbb{R}^N$ and $n^{(i)} \in \mathbb{R}^{N_i}$, $i = 2,\dots,r$.

Geometric Hardy inequality on the starshaped sets

Let Ω be a starshaped set on a Carnot group. Then for every $\gamma \in \mathbb{R}$ and $p > 1$ we have the following Hardy inequality

$$\int_\Omega |\nabla_H f(x)|^p dx \geq -(p-1)(|\gamma|^{\frac{p}{p-1}} + \gamma)\int_\Omega \frac{|\nabla_H\langle Z(x),n(x)\rangle|^p}{|\langle Z(x),n(x)\rangle|^p}|f(x)|^p dx \tag{4}$$

$$+\gamma\int_\Omega \frac{\mathcal{L}_p(\langle Z(x),n(x)\rangle)}{|\langle Z(x),n(x)\rangle|^{p-1}}|f(x)|^p dx,$$

for every function $f \in C_0^\infty(\Omega)$.

Geometric Hardy inequalities on $\mathbb{H}^*$ and $\mathbb{E}^*$

Let $\mathbb{H}^*$ and $\mathbb{E}^*$ be starshaped sets on the Heisenberg group $\mathbb{H}_1$ and the Engel group. Then for every function $f \in C_0^\infty(\mathbb{H}^*)$, we have the following Hardy inequalities

$$\int_{\mathbb{H}^*} |\nabla_H f(x)|^2 dx \geq \frac{1}{4}\int_{\mathbb{H}^*}\frac{|(n_1+4x_2n_3, n_2-4x_1n_3)|^2}{|x_1n_1+x_2n_2+2x_3n_3|^2}|f(x)|^2dx, \tag{5}$$

and for all $\gamma \in \mathbb{R}$, and $f \in C_0^\infty(\mathbb{E}^*)$,

$$\int_{\mathbb{E}^*} |\nabla_H f(x)|^2 dx \geq -(|\gamma|^2+\gamma)\int_{\mathbb{E}^*}\frac{|\nabla_H\langle Z(x),n(x)\rangle|^2}{\langle Z(x),n(x)\rangle^2}|f(x)|^2dx \tag{6}$$

$$+\frac{\gamma}{2}\int_{\mathbb{E}^*}\frac{x_2n_4}{\langle Z(x),n(x)\rangle}|f(x)|^2dx.$$

Hardy inequality on the half-spaces of M

Let us define the half-space of a sub-Riemannian manifold by

$$\Omega^+ := \{x \in \mathbb{R}^n : \langle x, n(x)\rangle > d\},$$

where $n = n(x) \in \mathbb{R}^n$ is the Riemannian outer unit normal to $\partial\Omega^+$ and $d \in \mathbb{R}$. The Euclidean distance to the boundary $\partial\Omega^+$ is denoted by $dist(x,\partial\Omega^+)$ and defined by

$$dist(x,\partial\Omega^+) := \langle x, n(x)\rangle - d.$$

Let M be a sub-Riemannian manifold, let $\Omega^+ \subset M$ be a half-space and let $X_1,\dots,X_N$ be the general vector fields. Then for every $\gamma \in \mathbb{R}$ and every $p > 1$, we have the following Hardy inequality

$$\int_{\Omega^+}|\nabla_X f|^p dx \geq -(p-1)(|\gamma|^{\frac{p}{p-1}}+\gamma)\int_{\Omega^+}\frac{|\nabla_X dist(x,\partial\Omega^+)|^p}{dist(x,\partial\Omega^+)^p}|f|^p dx$$

$$+\gamma\int_{\Omega^+}\frac{\mathcal{L}_p(dist(x,\partial\Omega^+))}{dist(x,\partial\Omega^+)^{p-1}}|f|^p dx,$$

for every function $f \in C_0^\infty(\Omega^+)$.
Note that above inequality was obtained for the Carnot groups by the authors in [3], but here we extend it to general sub-Riemannian manifolds.
Let Ω^+ be a half-space in the Grushin plane G. Then for every function $f \in C_0^\infty(\Omega^+)$ and every $p > 1$, we have the following Hardy inequality

$$\int_{\Omega^+}|\nabla_X f|^p dx \geq -(p-1)(|\gamma|^{\frac{p}{p-1}}+\gamma)\int_{\Omega^+}\frac{(n_1^2+x_1^2n_2^2)^{p/2}}{(x_1n_1+x_2n_2-d)^p}|f|^p dx$$

$$+(p-2)\gamma\int_{\Omega^+}\frac{|\nabla_X dist(x,\partial\Omega^+)|^{p-4}n_1n_2^2x_1}{(x_1n_1+x_2n_2-d)^{p-1}}|f|^p dx.$$

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